

OFF-SHELL HODGE DUALITIES IN LINEARISED GRAVITY AND E_{11}

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Abstract

In a spacetime of dimension n , the dual graviton is characterised by a Young diagram with two columns, the first of length $n-3$ and the second of length one. In this paper we perform the off-shell dualisation relating the dual graviton to the double-dual graviton, displaying the precise off-shell field content and gauge invariances. We then show that one can further perform infinitely many off-shell dualities, reformulating linearised gravity in an infinite number of equivalent actions. The actions require supplementary mixed-symmetry fields which are contained within the generalised Kac-Moody algebra E_{11} and are associated with null and imaginary roots.

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1 Introduction

The identification of symmetries of the low-energy limits of M-theory is expected to illuminate the complete definition of M-theory. Several proposals have been made that identify Kac-Moody algebras within supergravity. It was conjectured by West that the non-linear realisation of the generalised Kac-Moody algebra E_{11} is an extension of maximal supergravity relevant to M-theory [1]. Explicit constructions were also discovered that exhibited the hyperbolic Kac-Moody algebra E_{10} as a symmetry of the equations of motion in the vicinity of a cosmological singularity [2, 3]. That affine and hyperbolic Kac-Moody algebras would appear as hidden symmetries of supergravity, when the theory is dimensionally reduced, had been anticipated previously [4, 5].

The Kac-Moody algebra E_{11} may be decomposed into an infinite set of highest weight tensor representations of \mathfrak{sl}_{11} which are graded by the level within the decomposition at which they occur. At low levels, E_{11} contains a field which West identified with the dual graviton [1]. In a spacetime of dimension $n = 11$, the dual graviton is characterised by a Young diagram with two columns, the first of length $n - 3 = 8$ and the second of length one, which we denote by the symbol $C_{[8,1]}$ corresponding to the components $C_{\mu_1 \dots \mu_8, \nu} \equiv C_{\mu[8], \nu}$. In [1] a first-order action was proposed which is equivalent to nonlinear gravity and features an auxiliary field that, after extremising the action with respect to the vielbein and linearising the resulting equation, is identified with the curl of the dual graviton. It was later shown in [6] that the action given in [1], linearised and specifying $n = 5$, lead to the action proposed by Curtright in [7], and for arbitrary $n \geq 5$ to the action given by Aulakh et al. in [8]. In the following we will refer to the action for the dual graviton $S_{\text{Curt.}[C_{[n-3,1]}]}$ as the Curtright action while the Fierz–Pauli action $S_{\text{FP}}[h_{[1,1]}]$ [9] is equivalent to the Einstein–Hilbert action linearised around a Minkowski background of dimension n .

To reiterate, the action proposed in [1], when linearised¹, was shown [6] to reproduce not only the Fierz–Pauli action upon extremising with respect to the auxiliary field, but also the Curtright action after extremising with respect to the linearised vielbein and substituting the result inside the parent action, thereby demonstrating that the Fierz–Pauli and Curtright actions are equivalent to each other.

Here we consider all further dualisations of the graviton and first construct the action of the double-dual graviton $D_{\mu[n-3], \nu[n-3]}$, a field which was proposed in [12–14], whose parent action also reduces to the Curtright action upon algebraic elimination of one of its fields. The procedure will be described for an infinite set of further dualisations of the graviton giving rise to three infinite gravity

¹In order to have a nonlinear action principle equivalent to full gravity and containing at the same time the graviton and its dual (and not an extra field that reproduces on-shell the curl of the dual graviton in the linearisation), one needs to add extra fields. This was done in [10], following a procedure used in the context of gauged supergravity. Actually, a set of equations equivalent to the field equations derived in [10] had been proposed earlier in [11], albeit in a different and non-Lagrangian form, and where the extra field had been introduced with remarkable insight. The results obtained in [10] (where also the gauge structure of the theory was clarified), therefore strengthen and give an alternative way of understanding the equations found in [11].

towers with fields $\tilde{h}_{\mu_1[n-2],\dots,\mu_k[n-2],\nu,\rho}$, $\tilde{C}_{\mu_1[n-2],\dots,\mu_k[n-2],\nu[n-3],\rho}$ and $\tilde{D}_{\mu_1[n-2],\dots,\mu_k[n-2],\nu[n-3],\rho[n-3]}$ ($k = 1, 2, \dots$) which we refer to as the Fierz–Pauli tower, the dual graviton tower and the double-dual tower, respectively. We note that the fields entering what we call here the dual graviton tower were first recognised as an infinite set of dual gravitons in [15]. From the work in the present paper the reader will be able to construct the actions for the fields in any of the infinite towers.

It has been proposed by Hull [12–14] that the further off-shell dualisation of the Fierz–Pauli graviton, if it is possible, should unveil some hidden symmetries of M-theory that had gone unnoticed before. Hull conjectured a duality between an exotic six-dimensional $(4, 0)$ superconformal theory and the strong coupling limit of maximally supersymmetric $\mathcal{N} = 8$ supergravity in 5 dimensions. Upon dimensional reduction of the field content of the linearised six-dimensional theory down to five dimensions [12–14], not only does the dual graviton $\begin{smallmatrix} \square & \square \end{smallmatrix}$ appear but also a double-dual graviton $\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$. The exotic interacting six-dimensional theory suggested by Hull is to maximally $\mathcal{N}_5 = 8$ supergravity what the superconformal $\mathcal{N}_6 = (2, 0)$ theory is to maximally supersymmetric Yang–Mills theory in five dimensions. It is very tempting to think, like Hull, that there is a corner of M-theory that contains the exotic $\mathcal{N}_6 = (4, 0)$ theory. Note that this theory has been discussed recently in [16].

In the present paper we show that it is indeed possible to nontrivially dualise Curtright’s action, thereby making contact with Hull’s double-dual graviton. The off-shell and manifestly covariant formulation of the double-dual graviton that we obtain is less economical than Curtright’s dual formulation or the Fierz–Pauli original one, in the sense that the off-shell spectrum is larger, although on-shell the degrees of freedom are the same by construction. We show that it is actually possible to perform infinitely many further off-shell Hodge dualisations to obtain what we call here the Fierz–Pauli, dual and double-dual graviton towers, thereby describing linearised gravity in less and less economical formulations. These descriptions, on the other hand, allow us to make explicit contact with generators of E_{11} .

In [15] fields of exactly the same symmetry types as those entering the dual graviton tower, e.g. containing a Young tableau of the type $\tilde{C}_{\mu_1[n-2],\dots,\mu_k[n-2],\nu[n-3],\rho}$ for $n = 11$, were identified within the generalised Kac-Moody algebra E_{11} . It was conjectured therein that this infinite tower of \mathfrak{sl}_{11} representations contained all possible on-shell dual descriptions of the graviton. The work in this paper supports the interpretation that the tower of \mathfrak{sl}_{11} representations identified in [15] indeed contains dual gravitational fields and places it on a firm off-shell footing. Additionally we identify further infinite gravity towers which are not contained in E_{11} . We demonstrate how these dual fields may be incorporated at the level of an action via Hodge duality. The actions require sets of supplementary fields, described by mixed symmetry tensors, which are all contained in E_{11} and associated with null or imaginary roots.

The fields in what we call here the dual graviton tower have also been interpreted in [17] as exotic gravitational solutions where it was shown that each of these exotic gravity fields are related to each other by the Geroch group [18–20]. Additionally the dual graviton tower of fields has been argued [21]

in the context of E_{10} to be related to spatial derivatives of the field $C_{\mu[n-3],\nu}$.

In Section 2 we explicitly construct a parent action that on the one hand reduces to Curtright's action upon eliminating some auxiliary fields, and on the other hand produces a new action that features Hull's double-dual graviton upon eliminating another set of auxiliary fields. We analyse the gauge invariance of the aforementioned new action.

In Section 3 we show how infinitely many dual formulations of Fierz–Pauli theory can be obtained, which become less and less economical in the sense that the off-shell spectrum entering the successive Lagrangians gets bigger and bigger.

In Section 4 we relate the field content of the double-dual action of Section 2 containing the spectrum of the $(4, 0)$ linearised theory proposed by Hull with the generators of E_{11} . The gravitational degrees of freedom present in the five-dimensional theory are traced to their six-dimensional origin and the double-dual graviton of Hull is shown not to be present within E_{11} . The gravitational degrees of freedom are instead identified with an exotic gravity field within the dual tower of fields of E_{11} associated with the dual graviton. Furthermore the off-shell fields required for the action constructed in Section 3 are identified with the null and imaginary roots of E_{11} and appear at the same level in the algebraic decomposition as the exotic dual gravity field in question.

We conclude the paper with a summary of our work and some comments in Section 5.

2 On-shell and off-shell double dualisation

Preamble: on-shell dualisation. Before attacking the problem of off-shell dualisation of linearised gravity beyond the Curtright level, where the dual graviton appears, we first show that it is very simple to write a quadratic action that features both the double-dual field $D_{\mu[n-3],\nu[n-3]}$ and the Fierz–Pauli field $h_{\mu\nu}$ and produces, on-shell, the appropriate duality relation between them together with the Fierz–Pauli equations. Such a dualisation is however not satisfactory since it takes place only on-shell instead of off-shell, nevertheless it prepares the grounds for the rest of the paper and allows us to introduce our notation. Taking $n = 5$ for the sake of clarity, one considers the following action²

$$S[h_{\mu\nu}, D_{\mu\nu,\rho\sigma}] = \int d^5x \, \varepsilon^{\mu\nu\rho\sigma\lambda} \varepsilon_{\alpha\beta\gamma\delta\epsilon} D_{\mu\nu,}{}^{\alpha\beta} \left(R_{\rho\sigma,}{}^{\gamma\delta}(h) \delta_{\lambda}^{\epsilon} - \frac{1}{2} \partial_{\rho} \partial^{\gamma} D^{\delta\epsilon,}{}_{\sigma\lambda} \right) \quad (2.1)$$

where $R_{\mu\nu}{}^{\rho\sigma}(h) = 2 \partial^{[\rho} \partial_{[\mu} h_{\nu]}^{\sigma]}$ is the linearised Riemann tensor of the field $h_{\mu\nu}$. The gauge invariances read

$$\delta h_{\mu\nu} = 2 \partial_{(\mu} \epsilon_{\nu)} \quad , \quad \delta D_{\mu\nu,\rho\sigma} = (\partial_{\mu} \lambda_{\rho\sigma,\nu} - \partial_{\nu} \lambda_{\rho\sigma,\mu}) + (\partial_{\rho} \lambda_{\mu\nu,\sigma} - \partial_{\sigma} \lambda_{\mu\nu,\rho}) \quad (2.2)$$

²We choose the space-time signature to be $(- + \dots +)$. The epsilon symbol is defined by $\varepsilon^{012\dots} = +1$. We denote strength-one antisymmetrisation of indices by square brackets and the components of an antisymmetric tensor $T_{a_1\dots a_p} = T_{[a_1\dots a_p]}$ will sometimes be denoted by $T_{a_1\dots a_p} = T_{a[p]}$. Similarly, we sometimes use $\partial_{\mu} e_{\mu,\nu} \equiv \frac{1}{2} (\partial_{\mu_1} e_{\mu_2,\nu} - \partial_{\mu_2} e_{\mu_1,\nu})$. Differential form degree is denoted by a subscript in boldface font, so that $e_2^{a[2]}$ denotes a two-form taking its values in the antisymmetric rank-two irreducible representation of the Lorentz algebra. The (flat) background vielbein is denoted by \bar{h}_1^a .

where $\lambda_{\mu\nu,\rho}$ is an irreducible \mathfrak{gl}_5 tensor of type $[2, 1]$, *i.e.* it obeys $\lambda_{\mu\nu,\rho} = -\lambda_{\nu\mu,\rho}$, $\lambda_{[\mu\nu,\rho]} = 0$. As for the field equations, introducing the linearised curvature for the double dual graviton as $K_{\mu\nu\rho}{}^{\alpha\beta\gamma} = 12 \partial^{[\gamma} \partial_{[\rho} D_{\mu\nu]}{}^{\alpha\beta]}$, we have

$$\left\{ \begin{array}{l} \frac{\delta S[h, D]}{\delta D^{\mu\nu, \rho\sigma}} = 0 \\ \frac{\delta S[h, D]}{\delta h^{\mu\nu}} = 0 \end{array} \right. \iff \left\{ \begin{array}{l} R_{\mu\nu, \alpha\beta}(h) = \varepsilon_{\mu\nu\rho\sigma\lambda} K^{\rho\sigma\lambda, \gamma\delta\epsilon} \varepsilon_{\alpha\beta\gamma\delta\epsilon} \\ \eta^{\rho\gamma} \eta^{\sigma\delta} K_{\mu\rho\sigma, \nu\gamma\delta} = 0 \end{array} \right. , \quad (2.3)$$

which can be written in the hyperform notation [22] as $R_{[2,2]} = *_1 *_2 K_{[3,3]}$ and $\text{Tr}^2 K_{[3,3]} = 0$, or equivalently as $R_{[2,2]} = *_1 *_2 K_{[3,3]}$ and $\text{Tr} R_{[2,2]} = 0$ where the last equation is the Fierz–Pauli equation for the spin-2 field $h_{\mu\nu}$. The above equations were discussed by Hull in [12–14].

The action $S[h, D]$ (2.1) can be written in the frame-like formalism of [23–26]. Since the first-order action for the $D_{[2,2]}$ -field in five dimensions is (see e.g. Section 3 of [23])

$$S[e_2^{a[2]}, \omega_2^{a[3]}] = \int_{\mathcal{M}_5} \epsilon_{a[5]} \left(\frac{3}{8} \bar{h}_1^a \wedge \omega_2^{a[2]c} \wedge \omega_2^{a[2]}{}_c + \frac{1}{12} e_2^{a[2]} \wedge d\omega_2^{a[3]} \right) , \quad (2.4)$$

we can directly write down the action (2.1) in the following hybrid form:

$$S[h_1^a, e_2^{a[2]}, \omega_2^{a[3]}] = \int_{\mathcal{M}_5} \epsilon_{a[5]} \left(\frac{3}{8} \bar{h}_1^a \wedge \omega_2^{a[2]c} \wedge \omega_2^{a[2]}{}_c + \frac{1}{12} e_2^{a[2]} \wedge d \left[\omega_2^{a[3]} + \bar{h}_1^a \omega_1^{a[2]}(h_1) \right] \right) \quad (2.5)$$

where $\omega_1^{a[2]}(h_1) = dx^\mu \omega_\mu^{a[2]}(h_1)$ is the spin connection one-form viewed as a function of the dynamical vielbein fluctuation $h_1^a = dx^\mu h_\mu^a$ via the solution of the linearised zero-torsion $dh_1^a + \omega_1^{a b} \bar{h}_1^b = 0$ condition. The action (2.5) is truly first-order in the sector of the $D_{[2,2]}$ field. In the spin-2 sector, it features the spin-connection as a function as the vielbein fluctuation, as in the linearisation of the Einstein–Cartan–Weyl action [the Einstein–Cartan–Weyl action is recalled in (2.6) below]. The action (2.5) lends itself to non-linear completion where one replaces everywhere the background vielbein \bar{h} by $e_1^a = \bar{h}_1^a + h_1^a$ and $d\omega_1^{a[2]}(h_1)$ by the full non-linear curvature $R_1^{a[2]}(e_1) = d\omega_1^{a[2]}(e_1) + \omega_1^{ab}(e_1)\omega_{1b}{}^a(e_1)$.

Although the action proposed above has the advantage of relating the double-dual field introduced by Hull to the usual graviton, the dualisation relation is only obtained on-shell which is not sufficient for a genuine equivalence of theories and for the purpose of quantisation [27]. One needs a parent action that relates the Fierz–Pauli action to a new action that would incorporate the double-dual field $D_{a[n-3], b[n-3]}$. We construct the parent action in the sequel.

Off-shell dualisation: first round. We start by reviewing the off-shell dualisation of the graviton as given in [1, 6]. For this one uses the fact that the second-order Einstein–Hilbert action based on the vielbein $e_\mu{}^a$ can be written, up to boundary terms, as [28]

$$S_{\text{EH}}[e_{ab}] = - \int d^n x e \left(\Omega^{ab, c} \Omega_{ab, c} + 2 \Omega^{ab, c} \Omega_{ac, b} - 4 \Omega_{ab}{}^b \Omega^{ac}{}_c \right) , \quad (2.6)$$

where

$$\Omega_{ab,}{}^c = \Omega_{ab,}{}^c(e) = 2 e_a{}^\mu e_b{}^\nu \partial_{[\mu} e_{\nu]}{}^c \quad (2.7)$$

are the coefficients of anholonomicity. This form of the Einstein-Hilbert action can be recast into first-order form by introducing an auxiliary field $Y_{ab,c} = -Y_{ba,c}$,

$$S[Y_{ab,c}, e_{ab}] = -2 \int d^n x e \left(Y^{ab,c} \Omega_{ab,c}(e) - \frac{1}{2} Y_{ab,c} Y^{ac,b} + \frac{1}{2(n-2)} Y_{ab,}{}^b Y^{ac,}{}_c \right). \quad (2.8)$$

The field equation of Y can be used to solve for it in terms of $\Omega(e)$,

$$Y_{ab,c}(e) = \Omega_{ab,c} - 2\Omega_{c[a,b]} + 4\eta_{c[a}\Omega_{b]d,}{}^d. \quad (2.9)$$

After reinserting (2.9) into (2.8), one precisely recovers the Einstein-Hilbert action in the form (2.6). In fact, the action (2.8) coincides with the standard first order action with the spin connection as independent field, up to a mere field redefinition which replaces the spin connection by $Y_{ab|c}$. For later use one notes that (2.8) has the same symmetries as the original Einstein-Hilbert action. First, it is manifestly diffeomorphism invariant. Moreover, the invariance of the second-order action (2.6) under the local Lorentz group can be elevated to a symmetry of the first-order action by requiring that the auxiliary field $Y_{ab|c}$ transforms as

$$\delta_\Lambda Y_{ab,c} = -2 e_c{}^\mu \partial_\mu \Lambda_{ab} - 4\eta_{c[a} e^{\mu d} \partial_\mu \Lambda_{b]d} - 2\Lambda^d{}_{[a} Y_{b]d,c} + \Lambda^d{}_c Y_{ab,d}. \quad (2.10)$$

In order to obtain the dual graviton from (2.8) one has to consider the linearised theory and vary with respect to the metric. Before linearising, it turns out to be convenient to first rewrite the action in terms of the Hodge dual of $Y^{ab,c}$:

$$Y^{ab,c} = \frac{1}{(n-2)!} \epsilon^{abc_1 \dots c_{n-2}} Y_{c_1 \dots c_{n-2},}{}^c. \quad (2.11)$$

This yields

$$S[Y, e] = -\frac{2}{(n-2)!} \int d^n x e \left(\epsilon^{abc_1 \dots c_{n-2}} Y_{c_1 \dots c_{n-2},}{}^c \Omega_{ab,c} + \frac{n-3}{2(n-2)} Y^{c_1 \dots c_{n-2},b} Y_{c_1 \dots c_{n-2},b} \right. \\ \left. - \frac{n-2}{2} Y^{c_1 \dots c_{n-3}a,}{}_a Y_{c_1 \dots c_{n-3}b,}{}^b + \frac{1}{2} Y^{c_1 \dots c_{n-3}a,b} Y_{c_1 \dots c_{n-3}b,a} \right). \quad (2.12)$$

In the linearisation around flat space, $e_\mu{}^a = \delta_\mu{}^a + \kappa h_\mu{}^a$, one can ignore the distinction between flat and curved indices. In particular, one has now $\Omega_{\mu\nu,\rho} = 2 \partial_{[\mu} h_{\nu]\rho}$, where the field $h_{\mu\nu}$ still has no symmetry. The field equation for $h_{\mu\nu}$ is

$$\partial_{[\mu_1} Y_{\mu_2 \dots \mu_{n-1}],\nu} = 0. \quad (2.13)$$

The Poincaré lemma then implies that Y is the curl of a potential $C_{\mu_1 \dots \mu_{n-3},\nu}$ (the dual graviton), that is completely antisymmetric in its first $n-3$ indices but has no definite \mathfrak{gl}_n symmetry otherwise:

$$Y_{\mu_1 \dots \mu_{n-2},\nu} = \partial_{[\mu_1} C_{\mu_2 \dots \mu_{n-2}],\nu}. \quad (2.14)$$

Inserting this back into the linearisation of (2.12) yields a consistent quadratic action $S[C]$ for the dual graviton that is equivalent to the Curtright action [7].

Up to now, $C_{\mu_1 \dots \mu_{n-3}, \nu}$ as defined by (2.14) does not transform in any irreducible \mathfrak{gl}_n representation since Y does not possess any irreducible Young-diagram symmetry. However, one may check that, after inserting (2.14) into the linearisation of (2.12), the resulting action $S[C]$ is invariant under the following shift symmetry

$$\delta_\Lambda C_{\mu_1 \dots \mu_{n-3}, \nu} = -\Lambda_{\mu_1 \dots \mu_{n-3} \nu} , \quad (2.15)$$

with completely antisymmetric shift parameter. Therefore, the totally antisymmetric part of $C_{\mu_1 \dots \mu_{n-3}, \nu}$ can be gauge-fixed to zero, giving rise to the dual graviton with a $[n-3, 1]$ Young-diagram symmetry. In other words, in the action $S[C]$ the dual graviton appears in a way similar to the graviton in the second-order Weyl action (2.6). One can formulate a genuinely first-order action principle for arbitrary \mathfrak{gl}_n -irreducible mixed-symmetry gauge fields in flat background. This has been done by Skvortsov in [25]. The latter formulation is the analogue of the vielbein formalism of gravity (2.8) in which the Lorentz transformations act as Stückelberg transformations and where the spin-connection is viewed as an off-shell independent field.

Let us mention that, even though (2.8) and thus (2.12) are first-order formulations of *non-linear* Einstein gravity, the identification of the dual graviton in (2.14) is only possible in the linearisation, since in the full theory the integrability condition (2.13) is violated [1]. This is in agreement with the fact that there is no local, manifestly Lorentz-invariant and non-abelian self-interacting theory for the dual graviton [29, 30].

Towards a second off-shell dualisation. In order to address the problem of a further dualisation, it is useful to introduce the following quadratic parent action [6]:

$$S[\Omega_{ab,c}, Y_{abc,d}] = - \int d^n x \left(2 \Omega_{ab,c} \partial_d Y^{dab,c} + \Omega^{ab,c} \Omega_{ab,c} + 2 \Omega^{ab,c} \Omega_{ac,b} - 4 \Omega_{ab,}{}^b \Omega^{ac,}{}_c \right) . \quad (2.16)$$

The field $Y_{abc,d} = Y_{[abc],d}$ is a Lagrange multiplier for the constraint $\partial_{[d} \Omega_{ab],c} = 0$, implying $\Omega_{ab,c} = \partial_{[a} h_{b]c}$ where $h_{\mu\nu}$ has no definite symmetry in its two indices. Eliminating Y that way, one finds that the action (2.16) becomes the Einstein–Hilbert action (2.6) taken at quadratic order, namely the Fierz–Pauli action. On the other hand, $\Omega_{ab,c}$ is an auxiliary field and can be eliminated from the action (2.16) using its algebraic equations of motion. The resulting action is then the action (2.12) at quadratic order, in turn equivalent to the Curtright action.

The action (2.16) makes the first dualisation step transparent: Knowing the Fierz–Pauli action $S^{FP}[h_{ab}]$ given by the last three terms of (2.16) with $\Omega = \Omega(h)$, one introduces a new field $Y^{abd,c} =$

$Y^{[abd],c}$ that, via the first term $2 \int \Omega_{ab,c} \partial_d Y^{dab,c}$ enforces the relation $\Omega_{ab,c} = 2 \partial_{[a} h_{b]c}$ between the first connection of the spin-2 field and the spin-2 field itself in the frame formulation of (linearised) gravity where the vielbein has no definite symmetry in its indices. Then, eliminating the connection Ω from the action, one obtains a new action which is by construction equivalent to the Fierz–Pauli action and yields indeed the Curtright action in a formulation where the off-shell dual field $C_{\mu_1 \dots \mu_{n-3}, \nu}$ has no definite \mathfrak{gl}_n symmetry. One can then check that there is a shift symmetry (2.15) that ensures that only the \mathfrak{gl}_n -irreducible $[n-3, 1]$ part contributes to the action.

Accordingly, in order to understand a second dualisation at the level of the action, one has to follow the following procedure:

- (i) Construct the putative action:

$$S^{\text{put.}}[H^{a[n-3]},_{bc}, D^{bcd},_{a[n-3]}] = \int d^n x \left[H^{a[n-3]},_{bc} \partial_d D^{bcd},_{a[n-3]} + “HH” \right] \quad (2.17)$$

where the part denoted “ HH ” should give the Curtright action via the substitution $H_{\mu[n-3],}^{\nu[2]} \rightarrow 2 \partial^{[\nu_1} C_{\mu[n-3],}^{\nu_2]}$. In other words, a necessary condition is that the Curtright action admits a formulation $S_{\text{Curt.}}[H(C)]$ in which the \mathfrak{gl}_n -reducible field $C_{\mu[n-3], \nu}$ appears only through the quantity $H_{\mu[n-3],}^{\nu[2]}(C) := 2 \partial^{[\nu_1} C_{\mu[n-3],}^{\nu_2]}$;

- (ii) Then, supposing that (i) is possible, the field $D^{b[3]},_{a[n-3]}$ can be eliminated from the action (2.17), enforcing the relation $H_{\mu[n-3],}^{\nu[2]} = 2 \partial^{[\nu_1} C_{\mu[n-3],}^{\nu_2]}$ and thereby producing the Curtright action $S_{\text{Curt.}}[H(C)]$. Alternatively, one can extremise the action (2.17) with respect to the auxiliary field $H^{a[n-3]},_{b[2]}$ and get an action

$$S[D^{bcd},_{a[n-3]}] = \int d^n x \left[\partial^e D_{bce},_{a[n-3]} \partial_d D^{bcd},_{a[n-3]} + \dots \right] \quad (2.18)$$

which would, by construction, be equivalent to the Curtright action $S_{\text{Curt.}}[H(C)]$;

- (iii) Decomposing the D field into its irreducible \mathfrak{gl}_n components

$$D_{b[3],}^{a[n-3]} = X_{b[3],}^{a[n-3]} + Z_{b[3],}^{a[n-3]}, \quad (2.19)$$

$$Z_{b[3],}^{a[n-3]} := \delta_{[b_1}^{[a_1} Z_{b_2 b_3],}^{(1) a_2 \dots a_{n-3}} + \delta_{[b_1}^{[a_1} \delta_{b_2}^{a_2} Z_{b_3],}^{(2) a_3 \dots a_{n-3}} + \delta_{[b_1}^{[a_1} \delta_{b_2}^{a_2} \delta_{b_3]}^{a_3} Z^{(3) a_4 \dots a_{n-3}} \quad (2.20)$$

$$X_{b_1 b_2 b_3,}^{b_1 a[n-4]} \equiv 0 \equiv Z_{b_1 b_2,}^{(1) a[n-5]}, \quad Z_{b,}^{(2) a[n-6]} \equiv 0, \quad (2.21)$$

one obtains an action containing Hull’s two-column \mathfrak{gl}_n -irreducible gauge field

$$D_{a[n-3], b[n-3]} := \frac{1}{(n-3)!} \epsilon_{c[3] a[n-3]} X^{c[3]},_{b[n-3]}$$

(the ‘double-dual graviton’) provided that the components $Z_{b[3],}^{a[n-4]}$ of $D_{b[3],}^{a[n-3]}$ disappear from the action (2.18). If none of the Z components disappear from the action, one would get

an action equivalent to the Curtright action and expressed in terms of the set of \mathfrak{gl}_n -irreducible gauge fields

$$\{D_{a[n-3],b[n-3]}, E^{(1)}_{a[n-2],b[n-4]}, E^{(2)}_{a[n-1],b[n-5]}, Z^{(3)}_{b[n-6]}\},$$

where

$$E^{(1)}_{a[n-2],b[n-4]} := \frac{1}{(n-2)!} \epsilon_{c[2]a[n-2]} Z^{(1)c[2],b[n-4]}, \quad (2.22)$$

$$E^{(2)}_{a[n-1],b[n-5]} := \frac{1}{(n-1)!} \epsilon_{ca[n-1]} Z^{(2)c,b[n-5]}. \quad (2.23)$$

We now follow this programme and show that it is indeed possible to perform a second dualisation of the Fierz–Pauli action, the upshot being that all the following fields

$$\{D_{a[n-3],b[n-3]}, E^{(1)}_{a[n-2],b[n-4]}, E^{(2)}_{a[n-1],b[n-5]}, Z^{(3)}_{b[n-6]}\}, \quad (2.24)$$

enter the double-dual formulation. We will first achieve this programme in the case $n = 5$ simply in order to simplify the formulae and the presentation and then give the results in the general n -dimensional case, where $n \geq 5$. In the case $n = 4$, it is obvious that one keeps on reproducing Fierz–Pauli action, as was already explained in [6].

The first thing to do according to point (i) is to reformulate Curtright’s action in terms of the quantity $H_{\mu\nu}{}^{\rho\sigma}(C) = 2\partial^{[\rho}C_{\mu\nu}{}^{\sigma]}$. Taking into account that fact that the action should contain the kinetic term $\int d^5x \frac{1}{2} H_{\mu\nu}{}^{\rho\sigma}(C)H^{\mu\nu}{}_{\rho\sigma}(C)$ and should be invariant under the transformations

$$\delta C_{\mu\nu,\rho} = 2\partial_{[\mu}\xi_{\nu]\rho} + \frac{1}{2}\Lambda_{\mu\nu\rho}, \quad (2.25)$$

where $\xi_{\nu\rho}$ has no symmetry in its two indices while $\Lambda_{\mu\nu\rho} = \Lambda_{[\mu\nu\rho]}$, one can write all the possible quadratic terms in the action and fix the free coefficients in order to ensure the invariance under (2.25). The procedure is direct and gives the following result³

$$S_{\text{Curt.}}[H_{\mu\nu}{}^{\rho\sigma}(C)] = \int d^5x \left[\frac{1}{2} H_{\mu\nu}{}^{\rho\sigma} H^{\mu\nu}{}_{\rho\sigma} + H^{\mu\nu}{}_{\rho\sigma} H_{\mu}{}^{\rho}{}_{\nu}{}^{\sigma} - 3 H^{\mu\nu}{}_{\rho\nu} H_{\mu\sigma}{}^{\rho\sigma} \right. \\ \left. - H^{\mu\nu}{}_{\rho\nu} H^{\rho\sigma}{}_{\mu\sigma} + H_{\mu\nu}{}^{\mu\nu} H_{\rho\sigma}{}^{\rho\sigma} \right] = \int d^5x \mathcal{L}(H(C)). \quad (2.26)$$

Then, complying with point (ii) in the programme above, one considers the following parent action

$$S[D^{\rho\sigma\lambda}{}_{\mu\nu}, H^{\mu\nu}{}_{\rho\sigma}] = \int d^5x \left[-H^{\mu\nu}{}_{\rho\sigma} \partial_\lambda D^{\rho\sigma\lambda}{}_{\mu\nu} + \mathcal{L}(H) \right]. \quad (2.27)$$

³As a consistency check one can take this action, integrate by parts, where necessary, and show that it gives back the usual Curtright action up to boundary terms. This works indeed as expected. Notice that not all the possible terms bilinear in H have been used. The term $H^{\nu_1\nu_2,\rho_1\rho_2}H_{\rho_1\rho_2,\nu_1\nu_2}$, is omitted because it is redundant when $H = H(C)$.

This action is invariant under the gauge transformations given by

$$\delta H^{\mu\nu}{}_{\rho\sigma} = \partial_{[\rho}\Lambda^{\mu\nu}{}_{\sigma]} + 4\partial_{[\rho}\partial^{[\mu}\xi^{\nu]}{}_{\sigma]} \quad , \quad (2.28)$$

$$\delta D^{\mu\nu\rho}{}_{\sigma\tau} = -3\delta^{[\mu}{}_{[\sigma}\Lambda^{\nu\rho]}{}_{\tau]} - 12(\tfrac{3}{2}\delta^{[\mu}{}_{[\sigma}\partial_{\tau]}\xi^{\nu\rho]} - \tfrac{1}{2}\delta^{[\mu}{}_{[\sigma}\partial^{[\nu}\xi^{\rho]}{}_{\tau]} - \tfrac{3}{2}\delta^{[\mu}{}_{[\sigma}\partial^\nu\xi_{\tau]}{}^{\rho]} + \delta^{[\mu}{}_{[\sigma}\delta_{\tau]}^\nu\partial_\alpha\xi^{\rho]\alpha} - \delta^{[\mu}{}_{[\sigma}\delta_{\tau]}^\nu\partial^{[\rho]}\xi_{\alpha}{}^{\alpha]}) \quad . \quad (2.29)$$

It is easy to see that the gauge transformations for the D field are reducible. Indeed, $\delta_\lambda D^{\mu\nu\rho}{}_{\sigma\tau} = 0$ for

$$\delta\xi^{\mu\nu} = \partial^\mu\bar{\xi}^\nu \quad . \quad (2.30)$$

It is a consequence of analogous reducibilities for $\delta_\xi H$ and $\delta_\xi C$. In addition any $\delta_\xi H$ transformation with antisymmetric $\xi^{\mu\nu} = -\xi^{\nu\mu}$ can be absorbed by a redefinition of the Λ parameter. The same holds for $\delta_\xi D$. So the antisymmetric part of ξ can be redefined away.

Because the D field enters $S[D, H]$ only through the quantity $\partial_\lambda D^{\lambda\rho\sigma,\mu\nu}$ it is obvious that $S[D, H]$ is invariant under the additional gauge transformation

$$\delta_\psi D^{\rho\sigma\lambda}{}_{\mu\nu} = \partial_\tau\psi^{\rho\sigma\lambda\tau}{}_{\mu\nu} \quad , \quad (2.31)$$

where the differential gauge parameter ψ is totally antisymmetric in its two groups of indices separately.

Varying $S[D, H]$ with respect to $D^{\rho\sigma\lambda}{}_{\mu\nu}$ enforces the relation $H_{\mu\nu}{}^{\rho\sigma}(C) = 2\partial^{[\rho}C_{\mu\nu}{}^{\sigma]}$ that, when plugged back into (2.27), gives the action $S_{\text{Curt.}}[H_{\mu\nu}{}^{\rho\sigma}(C)]$. On the other hand, the variation of $S[D, H]$ with respect to the field $H_{\mu\nu}{}^{\rho\sigma}$ gives

$$\frac{\delta S[D, H]}{\delta H^{\mu\nu}{}_{\rho\sigma}} = -\partial_\lambda D^{\rho\sigma\lambda}{}_{\mu\nu} + H_{\mu\nu}{}^{\rho\sigma} + 2H_{[\mu}{}^{[\rho}{}_{\nu]}{}^{\sigma]} - 6\delta_{[\mu}^{[\rho}H_{\nu]\lambda}{}^{\sigma]\lambda} - 2\delta_{[\mu}^{[\rho}H^{\sigma]\lambda}{}_{\nu]\lambda} + 2\delta_{[\mu}^{[\rho}\delta_{\nu]}^{\sigma]}H^{\alpha\beta}{}_{\alpha\beta} \quad . \quad (2.32)$$

Solving the equation $\frac{\delta S[D, H]}{\delta H^{\mu\nu}{}_{\rho\sigma}} = 0$ for $H^{\mu\nu}{}_{\rho\sigma}$ gives

$$\begin{aligned} H_{\mu\nu}{}^{\rho\sigma} &= \tfrac{1}{2}(\partial_\lambda D^{\rho\sigma\lambda}{}_{\mu\nu} - \partial^\lambda D_{\mu\nu\lambda}{}^{\rho\sigma}) + \partial_\lambda D^{\lambda[\rho}{}_{[\mu}{}^{\sigma]}{}_{\nu]} - \tfrac{3}{2}\partial_\lambda D^{\lambda\alpha[\rho}{}_{\alpha[\mu}{}^{\sigma]}{}_{\nu]} \\ &\quad + \tfrac{1}{2}\partial^\lambda D_{\lambda\alpha[\mu}{}^{\alpha[\rho}{}_{\nu]}{}^{\sigma]} + 6\delta_{[\mu}^{[\rho}\delta_{\nu]}^{\sigma]}\partial_\lambda D^{\alpha\beta\lambda}{}_{\alpha\beta} \quad . \end{aligned} \quad (2.33)$$

Inserting this expression back into the action $S[D, H]$ yields

$$S[D^{\rho\sigma\lambda}{}_{\mu\nu}] = \tfrac{1}{4} \int d^5x \mathcal{L}(D) \quad , \quad (2.34)$$

where

$$\begin{aligned} \mathcal{L}(D) &= \left[-\partial_\lambda D^{\lambda\rho\sigma,\mu\nu}\partial^\alpha D_{\alpha\rho\sigma,\mu\nu} + \partial_\lambda D^{\lambda\rho\sigma,\mu\nu}\partial^\alpha D_{\alpha\mu\nu,\rho\sigma} - 2\partial_\lambda D^{\lambda\rho\sigma,\mu\nu}\partial^\alpha D_{\alpha\mu\rho,\nu\sigma} \right. \\ &\quad \left. + 3\partial_\lambda D^{\lambda\mu\sigma,\nu\rho}\partial^\alpha D_{\alpha\mu\rho,\nu\sigma} - \partial_\lambda D^{\lambda\mu\sigma,\nu\rho}\partial_\alpha D^{\alpha\nu\rho}{}_{\mu\rho} - \tfrac{1}{3}\partial_\lambda D^{\lambda\mu\nu}{}_{\mu\nu}\partial_\alpha D^{\alpha\rho\sigma}{}_{\rho\sigma} \right] \quad . \end{aligned} \quad (2.35)$$

We know that, by construction, the action $S[D^{\rho\sigma\lambda},_{\mu\nu}]$ is equivalent to Curtright's action $S_{\text{Curt.}}[C_{\mu\nu,\rho}]$ which in its turn is equivalent to the Fierz–Pauli action $S^{FP}[h_{\mu\nu}]$. The child action $S[D]$ inherits from its parent action $S[D, H]$ the invariance under the gauge transformations:

$$\begin{aligned}\delta_{\Lambda,\xi,\psi} D^{\mu\nu\rho},_{\sigma\tau} &= -3 \delta^{[\mu}_{[\sigma} \Lambda^{\nu\rho]}_{\tau]} + \partial_\lambda \psi^{\mu\nu\rho\lambda},_{\sigma\tau} \\ &\quad -12 \left(\frac{3}{2} \delta^{[\mu}_{[\sigma} \partial_\tau] \xi^{\nu\rho]} - \frac{1}{2} \delta^{[\mu}_{[\sigma} \partial^\nu \xi^{\rho]}_{\tau]} - \frac{3}{2} \delta^{[\mu}_{[\sigma} \partial^\nu \xi^{\rho]}_{\tau]} + \delta^{[\mu}_{[\sigma} \delta^\nu_{\tau]} \partial_\alpha \xi^{\rho]\alpha} - \delta^{[\mu}_{[\sigma} \delta^\nu_{\tau]} \partial^\rho \xi^\alpha_{\alpha]} \right).\end{aligned}\tag{2.36}$$

In order to pursue the last point (iii) of the above programme, we decompose the D field into its irreducible \mathfrak{gl}_n representations

$$D^{\mu\nu\rho},_{\sigma\tau} = X^{\mu\nu\rho},_{\sigma\tau} + \delta^{[\mu}_{[\sigma} Z^{(1)\nu\rho]},_{\tau]} + \delta^{[\mu}_{[\sigma} \delta^\nu_{\tau]} Z^{(2)\rho]}\tag{2.37}$$

and substitute this into the action $S[D]$. Contrary to what happens in the case of the Curtright action resulting from the off-shell dualisation of the Fierz–Pauli action, here we find that most of the components of the Z fields survive in the action. As one can see from (2.36), only one component of $Z^{(1)}$ disappears from the action (the totally antisymmetric component $Z^{(1)}_{[\nu\sigma,\tau]}$ which may be gauged away by $\Lambda_{\nu\rho\tau}$), and not all of it. In terms of the dual \mathfrak{gl}_5 -irreducible field $E^{(1)}_{\mu[3],\nu}$ corresponding to $Z^{(1)}$, (in n dimensions $E^{(1)}_{a[n-2],b[n-4]} := \frac{1}{(n-2)!} \epsilon_{c[2]a[n-2]} Z^{(1)c[2],b[n-4]}$), it means that only the traceless part of $E^{(1)}$ enters the action. The field $Z^{(2)}_\mu$ also survives inside the action. The remaining fields inherit the differential gauge transformations from (2.36). We now proceed to the $n \geq 5$ -dimensional construction.

General case with $n \geq 5$. We start from the Curtright action in dimension n , written in terms of $Y_{\mu[n-2],\nu}(C) = \partial_\mu C_{\mu[n-3],\nu}$:

$$S_{\text{Curt.}}[Y(C)] = \int d^n x \left[Y^{\lambda[n-2],\mu} Y_{\lambda[n-2],\mu} - \frac{(n-2)^2}{(n-3)} Y^{\lambda[n-3]\mu},_{\mu} Y_{\lambda[n-3]\nu},^{\nu} + \frac{(n-2)}{(n-3)} Y^{\lambda[n-3]\rho},_{\mu} Y_{\lambda[n-3]\mu},_{\rho} \right]\tag{2.38}$$

and rewrite this action in terms of the following object

$$H^{\lambda[n-3],}_{\mu\nu}(C) = \partial_{[\mu} C^{\lambda[n-3],}_{\nu]}.$$

A basis $\{A_i\}_{i=1,\dots,6}$ of all the possible terms entering the Lagrangian is given here:

$$\begin{aligned}A_1 &= H^{\lambda[n-3],\mu\nu} H_{\lambda[n-3],\mu\nu} \quad , & A_2 &= H^{\lambda[n-4]\nu,\rho\mu} H_{\lambda[n-4]\rho,\nu\mu} \quad , \\ A_3 &= H^{\lambda[n-5]\nu_1\nu_2,\rho_1\rho_2} H_{\lambda[n-5]\rho_1\rho_2,\nu_1\nu_2} \quad , & A_4 &= H^{\lambda[(n-4)]\nu},_{\nu}{}^{\mu} H_{\lambda[n-4]\sigma},^{\sigma}{}_{\mu} \quad , \\ A_5 &= H^{\lambda[n-5]\nu\sigma},_{\sigma}{}^{\rho} H_{\lambda[n-5]\rho\tau},^{\tau}{}_{\nu} \quad , & A_6 &= H^{\lambda[n-5]\nu\sigma},_{\nu\sigma} H_{\lambda[n-5]\rho\tau},^{\rho\tau} \quad .\end{aligned}\tag{2.39}$$

The resulting re-writing of Curtright's action reads

$$\begin{aligned}S_{\text{Curt.}}[H(C)] &= \int d^n x \left(\frac{2}{(n-2)} A_1 + \frac{4}{(n-2)} A_2 + \bar{\beta} A_3 \right. \\ &\quad \left. - 4 A_4 - \left[\frac{4(n-4)}{(n-2)} + 4\bar{\beta} \right] A_5 + \left[\frac{(n-4)(n-1)}{n-2} + \bar{\beta} \right] A_6 \right) \quad .\end{aligned}\tag{2.40}$$

Via the free parameter $\bar{\beta}$ there appears an ambiguity in the above form of the Lagrangian, due to the addition of total derivatives that modify the form of the Lagrangian but do not modify the action itself — in the present context we discard all boundary terms. The linear combination $I := A_3 - 4A_5 + A_6$ is a total divergence and hence does not contribute to the action. This ambiguity related to the addition of a total derivative to the Lagrangian will be reflected in a one-parameter ambiguity in the resulting dual action.

One can always rescale the action by an overall coefficient. After multiplying equation (2.40) by $\frac{n-2}{4}$ we obtain

$$\begin{aligned} S_{\text{Curt.}}[H(C)] &= \int d^n x \left[\frac{1}{2} A_1 + A_2 + \beta A_3 - (n-2)A_4 \right. \\ &\quad \left. - (n-4+4\beta)A_5 + \left[\frac{(n-1)(n-4)}{4} + \beta \right] A_6 \right] = \int d^n x \mathcal{L}^{\text{Curt.}}(H(C)) \end{aligned} \quad (2.41)$$

and we note that $\bar{\beta} = (\frac{4}{n-2})\beta$. One recovers the action (2.26) by setting $n = 5$ and $\beta = 0$.

At this stage we view the field $H^{\mu[n-3], \nu[2]}$ as independent and introduce a new field $D^{\mu[3], \nu[n-3]}$ leading to the following parent action

$$S[D^{\mu[3], \nu[n-3]}, H^{\mu[n-3], \nu[2]}] = \int d^n x \left[-H^{\mu[n-3], \nu[2]} \partial_\lambda D^{\lambda \nu[2], \mu[n-3]} + \mathcal{L}^{\text{Curt.}}(H) \right] \quad (2.42)$$

The parent action $S[D^{\mu[3], \nu[n-3]}, H^{\mu[n-3], \nu[2]}]$ is invariant under the following gauge transformations

$$\delta_{\Lambda, \xi} H^{\lambda[n-3], \nu \rho} = \partial_{[\nu} \Lambda^{\lambda[n-3]}_{\rho]} + \partial_{[\nu} \partial^\lambda \xi^{\lambda[n-4]}_{\rho]} \quad , \quad (2.43)$$

$$\delta_{\Lambda, \xi, \psi} D_{\mu \nu \rho}^{\lambda[n-3]} = 3(-1)^{n-4} (1-2\beta) \delta_{[\mu}^{\lambda} \Lambda_{\nu \rho]}^{\lambda[n-4]} + \delta_{\xi, \psi} D_{\mu \nu \rho}^{\lambda[n-3]} \quad , \quad (2.44)$$

where

$$\begin{aligned} \delta_{\xi, \psi} D^{\mu[3], \lambda[n-3]} &= 3 \left(\gamma_1 \delta_\lambda^\mu \partial^\mu \xi_{\lambda[n-5]}^{\mu, \lambda} + \gamma_2 \delta_\lambda^\mu \partial^\mu \xi_{\lambda[n-4]}^{\mu, \mu} + \gamma_3 \delta_\lambda^\mu \partial_\lambda \xi_{\lambda[n-5]}^{\mu, \mu} + \right. \\ &\quad \gamma_4 \delta_\lambda^\mu \partial_\lambda \xi_{\lambda[n-6]}^{\mu \mu, \lambda} + \gamma_5 \delta_\lambda^\mu \delta_\lambda^\mu \partial^\mu \xi_{\lambda[n-5]}^{\nu, \nu} + \gamma_6 \delta_\lambda^\mu \delta_\lambda^\mu \partial_\lambda \xi_{\lambda[n-6]}^{\mu, \rho} + \\ &\quad \gamma_7 \delta_\lambda^\mu \delta_\lambda^\mu \partial_\rho \xi_{\lambda[n-5]}^{\rho, \mu} + \gamma_8 \delta_\lambda^\mu \delta_\lambda^\mu \partial_\rho \xi_{\lambda[n-6]}^{\rho \mu, \lambda} + \gamma_9 \delta_\lambda^\mu \delta_\lambda^\mu \partial_\rho \xi_{\lambda[n-5]}^{\mu, \rho} + \\ &\quad \left. \gamma_{10} \delta_\lambda^\mu \delta_\lambda^\mu \partial_\rho \xi_{\lambda[n-6]}^{\rho, \nu} \right) + \partial_\nu \psi^{\mu[3] \nu, \lambda[n-3]} \quad , \end{aligned} \quad (2.45)$$

with

$$\begin{aligned} \gamma_1 &= \frac{4\beta+n-4}{n-3} \quad , \quad \gamma_2 = \frac{(n-2)}{(n-3)} \quad , \quad \gamma_5 = \frac{2(-1)^{n-3}}{n-3} \left[\frac{(n-4)(n-1)}{4} + \beta \right] \quad , \\ \gamma_7 &= -\frac{\gamma_3}{2} + (-1)^{n-4} \frac{(n-2)(n-4)}{2(n-3)} \quad , \quad \gamma_8 = -\gamma_4 + (-1)^{n-3} (n-4+4\beta) \frac{(n-5)}{2(n-3)} \quad , \\ \gamma_9 &= \frac{\gamma_3}{2} + (-1)^{n-4} \frac{(n-4+4\beta)}{2(n-3)} \quad , \quad \gamma_{10} = -\frac{\gamma_6}{3} + \frac{2}{3} \left[\frac{(n-4)(n-1)}{4} + \beta \right] \frac{(n-5)}{(n-3)} \quad . \end{aligned} \quad (2.46)$$

Apart from the parameter β , the other free parameters are $\{\gamma_3, \gamma_4, \gamma_6\}$. The freedom in the last three γ parameters reflects a redundancy between $\delta_\xi D_{\mu \nu \rho}^{\lambda[n-3]}$ and $\delta_\psi D_{\mu \nu \rho}^{\lambda[n-3]}$. Indeed, a ψ -transformation of the form

$$\psi^{\mu[4], \lambda[n-3]} = \theta_1 \delta_\lambda^\mu \delta_\lambda^\mu \xi_{\lambda[n-5]}^{\mu, \mu} + \theta_2 \delta_\lambda^\mu \delta_\lambda^\mu \xi_{\lambda[n-6]}^{\mu[2], \lambda} + \theta_3 \delta_\lambda^\mu \delta_\lambda^\mu \delta_\lambda^\mu \xi_{\lambda[n-6]}^{\mu \rho, \rho} \quad (2.47)$$

reproduces the 3-parameter freedom in $\{\gamma_3, \gamma_4, \gamma_6\}$, so that one may keep ψ arbitrary and set $\{\gamma_3, \gamma_4, \gamma_6\}$ to zero without loss of freedom in the gauge transformations.

Extremising the action (2.42) with respect to H gives the following relation:

$$\begin{aligned} \partial^\nu D_{\mu[2]\nu, \lambda[d-3]} &= H_1 + 2 H_2 + 2\beta H_3 - 2(n-2)(-1)^{n-4} H_4 \\ &\quad - 2(n-4+4\beta)(-1)^{n-4} H_5 + 2\left[\frac{(n-1)(n-4)}{4} + \beta\right] H_6 \quad , \end{aligned} \quad (2.48)$$

where

$$\begin{aligned} H_1 &= H_{\lambda[n-3], \mu[2]}, \quad H_2 = H_{\lambda[n-4]\mu, \lambda\mu}, \quad H_3 = H_{\lambda[n-5]\mu[2], \lambda\lambda} \quad , \\ H_4 &= \eta_{\lambda\mu} H_{\lambda[n-4]\sigma, \sigma\mu}, \quad H_5 = \eta_{\lambda\mu} H_{\lambda[n-5]\mu\tau, \tau\lambda}, \quad H_6 = \eta_{\lambda\mu} \eta_{\lambda\mu} H_{\lambda[n-5]\rho\tau, \rho\tau} \quad . \end{aligned} \quad (2.49)$$

with the convention that similar indices are implicitly antisymmetrised. The next step amounts to inverting the equation (2.48) in order to express the H field in terms of

$$T_{\lambda[n-3], \mu[2]} := \partial^\nu D_{\mu[2]\nu, \lambda[n-3]} \quad . \quad (2.50)$$

Having done that, one can replace the resulting expression $H(T)$ inside the parent action in order to obtain a “child action” $S[T(D)]$. After lengthy, but straightforward, computation introducing

$$\begin{aligned} T_1 &= T^{\lambda[n-3], \mu[2]} T_{\lambda[n-3], \mu[2]} \quad , \quad T_2 = T^{\lambda[n-4]\nu, \rho\mu} T_{\lambda[n-4]\rho, \nu\mu} \quad , \\ T_3 &= T^{\lambda[n-5]\nu_1\nu_2, \rho_1\rho_2} T_{\lambda[n-5]\rho_1\rho_2, \nu_1\nu_2} \quad , \quad T_4 = T^{\lambda[n-4]\nu, \nu\mu} T_{\lambda[n-4]\sigma, \sigma\mu} \quad , \\ T_5 &= T^{\lambda[n-5]\nu\sigma, \sigma\rho} T_{\lambda[n-3]\rho\tau, \tau\nu} \quad , \quad T_6 = T^{\lambda[n-5]\nu\sigma, \nu\sigma} T_{\lambda[n-5]\rho\tau, \rho\tau} \quad , \end{aligned} \quad (2.51)$$

we find

$$S[T(D)] = -\frac{1}{2} \int d^m x \quad (a_1 T_1 + a_2 T_2 + a_3 T_3 + a_4 T_4 + a_5 T_5 + a_6 T_6) \quad (2.52)$$

where

$$\begin{aligned} a_1 &= \frac{2(n-4)(n-4-\beta(n-5))}{(1-2\beta)((n-4)(n-1)+4\beta)} \quad , \quad a_2 = -\frac{(n-3)((n-4)(n-7)+4\beta(n-5))}{(1-2\beta)((n-4)(n-1)+4\beta)} \quad , \\ a_3 &= -\frac{(n-3)(n-4)(2\beta+n-4)}{(1-2\beta)((n-4)(n-1)+4\beta)} \quad , \quad a_4 = -\frac{(n-3)^2(3(n-4)-4\beta(n-5))}{2(1-2\beta)((n-4)(n-1)+4\beta)} \quad , \\ a_5 &= \frac{(n-3)^2(n-4)(n-4+4\beta)}{2(1-2\beta)((n-4)(n-1)+4\beta)} \quad , \quad a_6 = \frac{(n-3)^2(n-4)^2}{6((n-4)(n-1)+4\beta)} \quad . \end{aligned} \quad (2.53)$$

Setting $n = 5$ and $\beta = 0$ reproduces the action (2.34)-(2.35). The coefficients $\{a_i\}_{i=1,\dots,6}$ all have the same denominator, so one can multiply the action $S[T(D)]$ by an overall coefficient and thereby simplify the expression for the coefficients $\{a_i\}_{i=1,\dots,6}$. Notice that there are singular values for β , *i.e.* the denominators in (2.53) vanish for $\beta^{(1)} = 1/2$ and $\beta^{(2)} = \frac{-(n-4)(n-1)}{4}$. These values have to be rejected since they lead to a noninvertibility in the relation between H and T . More precisely, from (2.44) and $\beta = \beta^{(1)}$ one can see that the D field loses its algebraic gauge symmetry resulting in extra propagating degrees in freedom in D compared to $H(C)$. From (2.48)-(2.49), one sees that setting

$\beta = \beta^{(2)}$ removes the double-trace terms H_6 from the expression for the divergence of D in terms of H . We note that the parameter β does not exist in $n = 4$ as there is no suitable total derivative term that may be added to the Fierz–Pauli Lagrangian, the latter being fixed unambiguously from the requirement of gauge invariance.

Before closing this section, we would like to express the dual action $S[T(D)]$ in terms of the Hodge dual of T , introducing

$$\begin{aligned}\varepsilon^{\mu[2]\nu[n-2]} U^{\lambda[n-3],\mu[2]}_{\nu[n-2]} &= T^{\lambda[n-3],\mu[2]} \quad , \quad \text{where} \\ U^{\lambda[n-3],\mu[2]}_{\nu[n-2]} &= \partial_\nu \tilde{Y}_{\nu[n-3],\lambda[n-3]}^{\mu[2]} \quad \text{and} \\ \tilde{Y}_{\nu[n-3],\lambda[n-3]}^{\mu[2]} &= \frac{(-1)^{n-2}}{3!(n-3)!} \varepsilon_{\nu[n-3]\mu[3]} D^{\mu[3],\lambda[n-3]} \quad .\end{aligned}\tag{2.54}$$

This leads to the substitution of each term in (2.52) by a corresponding group of bilinear terms in U , constructed analogously to the T_i , $i = 1, \dots, 6$. Explicitly,

$$\begin{aligned}U_1 &= U^{\lambda[n-3],\mu[n-2]} U_{\lambda[n-3],\mu[n-2]} \quad , \quad U_2 = U^{\lambda[n-4]\nu,\rho\mu[n-3]} U_{\lambda[n-4]\rho,\nu\mu[n-3]} \quad , \\ U_3 &= U^{\lambda[n-5]\nu[2],\rho[2]\mu[n-4]} U_{\lambda[n-5]\rho[2],\nu[2]\mu[n-4]} \quad , \quad U_4 = U^{\lambda[n-4]\nu,\mu[n-3]} U_{\lambda[n-4]\sigma,\mu[n-3]}^\sigma \quad , \\ U_5 &= U^{\lambda[n-5]\nu\sigma,\sigma\rho\mu[n-4]} U_{\lambda[n-3]\rho\tau,\nu\mu[n-4]}^\tau \quad , \quad U_6 = U^{\lambda[n-5]\nu\sigma,\nu\sigma\mu[n-4]} U_{\lambda[n-5]\rho\tau,\rho\tau\mu[n-4]} \quad .\end{aligned}\tag{2.55}$$

The Hodge dualisation between the tensors T and U produces the following transformation at the level of the bilinear terms T_i and U_i , $i = 1, \dots, 6$:

$$\begin{aligned}T_1 &\rightarrow 2(n-2)! U_1 \quad , \quad T_2 \rightarrow (n-2)! [U_1 - (n-2)U_4] \quad , \\ T_3 &\rightarrow (n-2)! [2U_1 - 4(n-2)U_4 + (n-2)(n-3)U_6] \quad , \quad T_4 \rightarrow (n-2)! [U_1 - (n-2)U_2] \quad , \\ T_5 &\rightarrow (n-2)! [5U_1 - (n-2)U_4 - (n-2)U_2 + (d-2)(d-3)U_5] \quad , \\ T_6 &\rightarrow (n-2)! [2U_1 - 4(n-2)U_2 + (n-2)(n-3)U_3] \quad .\end{aligned}\tag{2.56}$$

Up to an overall normalisation with arbitrary parameter α , the result is

$$\begin{aligned}S[U(\tilde{Y})] &= \alpha \int d^n x \left\{ \left[\frac{(5n^3 - 77n^2 + 387n - 627)}{6(n-3)} - \frac{2(n^4 - 20n^3 + 154n^2 - 528n + 669)}{3(n-4)(n-3)} \beta \right] U_1 \right. \\ &\quad + \left[-\frac{(n-3)(n-2)(7n-37)}{6} + \frac{2(n-3)(n-2)(2n^2 - 22n + 59)}{3(n-4)} \beta \right] U_2 \\ &\quad + \left[\frac{(n-4)(n-3)^2(n-2)}{6} - \frac{(n-4)(n-3)^2(n-2)}{3} \beta \right] U_3 \\ &\quad + \left[-\frac{(n-2)(n^2 - 17n + 58)}{2} - \frac{2(n-2)(n^2 - 13n + 38)}{n-4} \beta \right] U_4 \\ &\quad + \left[\frac{(n-4)(n-3)^2(n-2)}{2} + 2(n-3)^2(n-2)\beta \right] U_5 \\ &\quad \left. + \left[-(n-4)(n-3)(n-2) - 2(n-3)(n-2)\beta \right] U_6 \right\} \quad ,\end{aligned}\tag{2.57}$$

where we have made explicit the freedom in overall normalisation with coefficient α . The action is valid for $n > 4$. For $n = 4$ all the procedure can be reproduced and as expected we find, up to a

rescaling, the Fierz–Pauli action (without a β parameter) where only the terms in U_1 , U_2 and U_4 remain.

The following change of variables slightly simplifies the result:

$$\begin{aligned} \alpha &= \frac{B+2\frac{A}{n-4}}{(n-3)(n-2)}, \quad \alpha\beta = \frac{B-A(n-1)}{2(n-3)(n-2)} \quad , \quad i.e. \\ A &= \alpha(1-2\beta)(n-4), \quad B = 4\alpha \left[\frac{(n-4)(n-1)}{4} + \beta \right] \quad . \end{aligned} \quad (2.58)$$

The variables A and B are chosen because they appear in the denominators of (2.53). In terms of these variables the action (2.57) acquires the following form

$$\begin{aligned} S[U(\tilde{Y})] &= \int d^n x \left[\left(\frac{n^2+11n-36}{3(n-2)} A + \frac{(n-5)(n^2-11n+26)}{2(n-4)(n-3)(n-2)} B \right) U_1 \right. \\ &\quad + \left(-\frac{2}{3} (n-6)(n-2)A - \frac{(n-5)(n-2)}{2(n-4)} B \right) U_2 \\ &\quad + \left(\frac{1}{6} (n-3)^2(n-2)A \right) U_3 \\ &\quad + \left((n-8)A - \frac{n^2-16n+52}{2(n-4)} B \right) U_4 + \\ &\quad + \left(-(n-3)(n-2)A + \frac{1}{2}(n-3)(n-2)B \right) U_5 + \\ &\quad \left. + ((n-3)A - (n-3)B) U_6 \right] \quad . \end{aligned} \quad (2.59)$$

3 Infinitely many off-shell dualisations

We have seen that the double-dual formulation of Fierz–Pauli gravity is less economical than the original formulation or than Curtright’s formulation in the sense that a larger spectrum of fields is needed for the manifestly covariant and local action.

In this section we show that one can actually describe linearised gravity around a flat background in infinitely many dual ways, each being manifestly Poincaré covariant and local, but featuring more and more fields.

Tower based on the Fierz–Pauli field. Starting from the Fierz–Pauli action

$$S[h_{[1,1]}] = \int d^n x \, L^{\text{FP}}(\partial_\alpha h_{\mu,\nu}) = \int d^n x \, \partial^\alpha h^{\mu,\nu} \partial_\alpha h_{\mu,\nu} + \dots \quad , \quad (3.60)$$

where $h_{[1,1]} \sim \square \otimes \square$, one introduces the independent field $G_1^{\alpha,\mu,\nu}$ which transforms in the representation $\square \otimes \square \otimes \square$ of \mathfrak{gl}_n contrary to the curl $\Omega \sim \partial_{[\alpha} h_{\mu],\nu} \sim \square \otimes \square$ that enters the linearisation of the action (2.6) and from which one arrives at the Curtright action via off-shell Hodge duality. One then writes the parent action

$$S_{\text{FP}}^{(P1)}[G_1, F_1] = \int d^n x \, \left(G_{\alpha,\mu,\nu} \partial_\beta F^{\beta\alpha,\mu,\nu} - \frac{1}{2} L^{\text{FP}}(G_1) \right) \quad , \quad (3.61)$$

where $F_1 \sim \square \otimes \square \otimes \square$.

Repeating the procedure used in the previous sections, from that parent action one either reproduces the Fierz–Pauli action $S_{\text{FP}}[h_{[1,1]}]$ upon extremising with respect to F_1 or another equivalent action

$$S_{\text{FP}}^{(1)}[h_{[n-2,1,1]}^{(1)}] = \int d^n x \left[\partial_{[\mu} h^{(1)}_{\mu[n-2]],\nu,\rho} \partial^{[\mu} h^{(1)\mu[n-2]],\nu,\rho} + \dots \right] \quad , \quad (3.62)$$

expressed in terms of the field $h_{[n-2,1,1]}^{(1)}$ obtained by Hodge dualising F_1 on the first column. For example, in dimension $n = 5$ the action $S_{\text{FP}}^{(1)}$ will feature the reducible field $h_{[3,1,1]}^{(1)}$ that decomposes under \mathfrak{gl}_5 into the following fields

$$\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \otimes \square \otimes \square \sim \underbrace{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}}_{\tilde{h}^{(1)}} \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \oplus 2 \times \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \quad . \quad (3.63)$$

Several comments are in order.

Firstly, the child action $S_{\text{FP}}^{(1)}$ inherits gauge invariances from its parent. The latter possesses an extension of the gauge invariances of the original Fierz–Pauli action. In particular, the field $h_{[n-2,1,1]}^{(1)}$ will be invariant under an algebraic gauge transformation containing an antisymmetric rank-2 tensor. In other words, a small set of fields in (3.63) will be gauged away.

Secondly, by construction we know that the child action $S_{\text{FP}}^{(1)}$ propagates the same physical on-shell degrees of freedom as the original Fierz–Pauli action, and we anticipate, drawing from our experience with Curtright’s action and with the double-dual action, that the on-shell field, in the light-cone gauge, will be given by the first field entering the decomposition of (3.63) as

$$\tilde{h}_{i[n-2],k,l}^{(1)} \approx \epsilon_{i[n-2]} h_{kl} \quad , \quad (3.64)$$

where h_{kl} is the \mathfrak{so}_{n-2} on-shell physical graviton. Clearly, $\tilde{h}_{i[n-2],k,l}^{(1)}$ is not traceless, and in this sense is not similar to h_{kl} . But although the field $\tilde{h}_{i[n-2],k,l}^{(1)}$ is not traceless, it is nevertheless non-identically vanishing and is propagating. The field $\tilde{h}_{i[n-2],k,l}^{(1)}$ transforms in exactly the same irreducible \mathfrak{so}_{n-2} representation as does h_{kl} , namely the spin-2 representation, and therefore gives yet another dual formulation of the graviton, like Curtright’s and the double-dual formulations in n dimensions that we have examined previously.

Analogously, we anticipate that further dual off-shell formulations of Fierz–Pauli theory will be given by an infinite number of actions $S^{(m)}[h_{[n-2,n-2,\dots,n-2,1,1]}^{(m)}]$ for $m = 2, 3, \dots$, where the gauge field $h^{(m)}$ possesses m sets of $n - 2$ antisymmetric indices on top of the two indices μ, ν carried by the original Fierz–Pauli field $h_{\mu,\nu}$. In particular, the field will contain the \mathfrak{gl}_n -irreducible component with the following symmetry type

$$\tilde{h}^{(m)} \sim \begin{array}{|c|c|c|} \hline n & n & \dots & n & n & n \\ \hline n-1 & n-1 & \dots & n-1 & & \\ \hline \vdots & \vdots & \dots & \vdots & & \\ \hline \end{array} \quad . \quad (3.65)$$

$$\begin{array}{|c|c|c|} \hline 4 & 4 & \dots & 4 \\ \hline 3 & 3 & \dots & 3 \\ \hline \end{array}$$

In order to see this, one starts from the resulting child action $S_{\text{FP}}^{(1)}[h_{[n-2,1,1]}^{(1)}]$, and notes that the basic object entering the Lagrangian is the gradient of $h^{(1)}$ and not its curl on the first column (integrating by parts can “undo” the anti-symmetrisations appearing in the curl). Denote the resulting gradient by the symbol G_2 with the symmetry type $[n-2] \otimes [1] \otimes [1] \otimes [1]$. A parent action is then obtained which features G_2 viewed as an independent field together with a new field F_2 with the symmetry type $[n-2] \otimes [2] \otimes [1] \otimes [1]$. extremising the parent action with respect to G_2 and substituting the solution of the resulting algebraic equation inside the parent action will produce the child action $S_{\text{FP}}^{(2)}[h_{[n-2,n-2,1,1]}^{(2)}]$ in terms of the gauge field $h_{[n-2,n-2,1,1]}^{(2)}$ obtained from F_2 by Hodge dualising the second column. Again, on-shell, the physical degrees of freedom will be carried by the component $\tilde{h}_{i[n-2],j[n-2],k,l}^{(2)}$ equivalent to h_{kl} via the relation

$$\tilde{h}_{i[n-2],j[n-2],k,l}^{(2)} \propto \epsilon_{i[n-2]} \epsilon_{j[n-2]} h_{kl} \quad . \quad (3.66)$$

Again, the resulting action $S_{\text{FP}}^{(2)}[h_{[n-2,n-2,1,1]}^{(2)}]$ can be dualised to give $S_{\text{FP}}^{(3)}[h_{[n-2,n-2,n-2,1,1]}^{(3)}]$ and so on and so forth, each one containing the \mathfrak{gl}_n -irreducible field $\tilde{h}^{(m)}$ depicted in (3.65), for $m = 1, 2, 3, \dots$

Dual graviton tower. In exactly the same way as we did starting from Fierz–Pauli’s action, one can now start from Curtright’s action and produce the tower of Hodge-dual actions $S_{\text{Curt.}}^{(m)}[C_{[n-2,\dots,n-2,n-3,1]}^{(m)}]$ that will each propagate the gauge field $\tilde{C}_{[n-2,\dots,n-2,n-3,1]}^{(m)}$ with \mathfrak{gl}_n -irreducible symmetry depicted as follows:

$$\tilde{C}^{(m)} \quad \sim \quad \begin{array}{ccc} \boxed{n} & \boxed{n} & \dots & \boxed{n} & \boxed{n} & \boxed{n} \\ \boxed{n-1} & \boxed{n-1} & \dots & \boxed{n-1} & \boxed{n-1} & \\ \vdots & \vdots & \dots & \vdots & \vdots & \\ \boxed{4} & \boxed{4} & \dots & \boxed{4} & \boxed{4} & \\ \boxed{3} & \boxed{3} & \dots & \boxed{3} & & \end{array} \quad . \quad (3.67)$$

where the number of columns with length $(n-2)$ is m .

On-shell, all these fields will be equivalent to the Curtright field, which is itself equivalent to the Fierz–Pauli field in the appropriate spacetime dimension n . In other words, all these fields, on-shell, transform in the spin-2 representation $\square\square$ of \mathfrak{so}_{n-2} . The corresponding actions $S_{\text{Curt.}}^{(m)}$ with $m = 1, 2, \dots$ give all different dual formulations of the same Fierz–Pauli action.

In the case $m = 1$ in five dimensions, the off-shell field is $C_{[3,2,1]}^{(1)}$ and decomposes under \mathfrak{gl}_5 into the following fields

$$\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \otimes \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \otimes \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \sim \underbrace{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}}_{\tilde{C}^{(1)}} \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \oplus 2 \times \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \oplus 2 \times \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \quad . \quad (3.68)$$

The double-dual’s tower. Finally, the same analysis can be done based on the double-dual action given in Section 2 to produce the tower of dual actions $S_{\text{dd}}^{(m)}[D^{(m)}]$ with $m = 1, 2, \dots$ that will each

propagate the gauge field $\tilde{D}_{[n-2, \dots, n-2, n-3, n-3]}^{(m)}$ with \mathfrak{gl}_n -irreducible symmetry depicted as follows:

$$\tilde{D}^{(m)} \sim \begin{array}{c} \begin{array}{|c|c|} \hline n & n \\ \hline n-1 & n-1 \\ \hline \vdots & \vdots \\ \hline 4 & 4 \\ \hline 3 & 3 \\ \hline \end{array} \dots \begin{array}{|c|c|c|} \hline n & n & n \\ \hline n-1 & n-1 & n-1 \\ \hline \vdots & \vdots & \vdots \\ \hline 4 & 4 & 4 \\ \hline 3 & & \\ \hline \end{array} \end{array} . \quad (3.69)$$

where the number of columns with length $(n-2)$ is m .

4 Maximal Supergravity, Dual Gravity and E_{11}

In the previous sections we have constructed manifestly, and in outline, the actions for an infinite set of dual formulations of linearised gravity. These dual formulations and indeed the considerations that underly their construction form part of the striking story of E_{11} , the conjectured symmetry algebra of M-theory [1]. The dual graviton tower of fields contained within E_{11} and argued to be dual descriptions of gravity in [15] have been shown to be equivalent to linearised gravity at the level of the action. In addition to the fields in the dual graviton tower a set of supplementary mixed-symmetry fields will appear in the action, see equation (3.67) where the five-dimensional supplementary fields are shown for the first field in the dual graviton tower. We will show that all the fields required to construct the actions for each of the individual fields entering the dual graviton tower are all contained within E_{11} and are associated with null and imaginary roots.

The work in this paper was inspired, in part, by the work of Hull on the double-dual graviton [12–14] and we commence this section with a search of the fields of E_{11} seeking the double-dual graviton. The double-dual graviton was identified by Hull [13] within the strongly coupled sector of five-dimensional maximally supersymmetric supergravity. In this section we will identify within the low levels of E_{11} the bosonic multiplets of maximal supergravity in five dimensions and the lift of these degrees of freedom into the $n = 6$ multiplets again contained within E_{11} . The multiplets of maximal supergravity in five and six dimensions derived from n and $n - 1$ forms have been found using E_{11} in [31, 32] and using the very extension of real forms of E_8 in [33]. We will see that it is not possible to identify in a straightforward way the six dimensional $(4, 0)$ multiplets within E_{11} that were originally found in [34] and which include the double-dual graviton. However each dual graviton in the dual gravity tower of fields carries the same number of degrees of freedom as the double-dual graviton and these fields do appear naturally within the decomposition of E_{11} together with the supplementary fields required to construct the actions described in this paper.

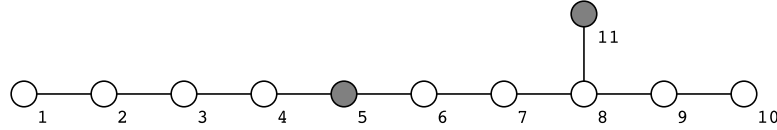
4.1 $\mathcal{N}_5 = 8$ maximal supergravity

The maximal supergravity in 5D, having $\mathcal{N} = 8$, may be decomposed into representations of the little group in 5D $Spin(3)$ and representations of $Sp(4)$, which is the local symmetry of the discrete

U-duality group $\frac{E_6}{Sp(4)}$. The on-shell multiplet splits into

$$(\mathbf{1}, \mathbf{42}) \oplus (\mathbf{2}, \mathbf{48}) \oplus (\mathbf{3}, \mathbf{27}) \oplus (\mathbf{4}, \mathbf{8}) \oplus (\mathbf{5}, \mathbf{1}) \quad (4.70)$$

giving 2^8 degrees of freedom. The bosonic degrees of freedom are given by **42** scalars, **27** vectors and **1** graviton (a symmetric 2-tensor, or bivector, field) in five dimensions. The decomposition of E_{11} gauge fields at low levels quickly identifies these U-duality multiplets for the bosonic fields. Consider the Dynkin diagram for E_{11}



where the shaded nodes indicate the decomposition relevant to the five-dimensional theory. The nodes $1, \dots, 4$ make up the Dynkin diagram of A_4 , or \mathfrak{sl}_5 , whose non-compact sub-group $SO(1, 4)$ will be the local Lorentz group for the five-dimensional spacetime. The remaining nodes $6, \dots, 11$ give the Dynkin diagram of E_6 , which contains the U-duality group in five dimensions and we will refer to this as the internal symmetry in five dimensions.

The positive roots of E_{11} may be written as a sum of the simple positive roots $\vec{\alpha}_i$ for $i = 1, 2, \dots, 11$ and will have the generic form:

$$\vec{\beta} = \sum_{i=1}^{11} m_i \vec{\alpha}_i. \quad (4.71)$$

The simple root associated with node 5 may be split into a vector in the A_4 weight lattice, a vector in the E_6 weight lattice and a part which is orthogonal to the fundamental weights of both A_4 and E_6 . We have

$$\alpha_5 = -\lambda_4 + x - \nu_6 \quad (4.72)$$

where λ_i for $i \in \{1, 2, 3, 4\}$ are the four fundamental weights of A_4 (indicated by nodes 1 to 4), ν_I for $I \in \{6, 7, \dots, 11\}$ is a fundamental weight of E_6 (nodes 6 to 11) and x is a vector in the weight lattice of E_{11} but orthogonal to the weight lattices of A_4 and E_6 . The decomposition of α_5 in equation (4.72) guarantees that the inner products of the simple roots of E_{11} are preserved, *i.e.* $\langle \alpha_5, \alpha_i \rangle = -\delta_{i4}$, $\langle \alpha_5, \alpha_I \rangle = -\delta_{I6}$ and $\langle \alpha_5, \alpha_5 \rangle = 2$. The last condition is used to normalise x .

Deletion of node 5 of the E_{11} Dynkin diagram splits the roots of E_{11} into a lowest weight representation of A_4 with Dynkin labels p_i ,

$$-\sum_{i=1}^4 p_i \lambda_i = -m_5 \lambda_4 + \sum_{i=1}^4 m_i \alpha_i \quad (4.73)$$

and a lowest weight representation of E_6 with Dynkin labels q_I

$$-\sum_{I=6}^{11} q_I \nu_I = -m_5 \nu_6 + \sum_{I=6}^{11} m_I \alpha_I. \quad (4.74)$$

The coefficients p_i and q_I label lowest weight representations of A_4 and E_6 respectively. It is useful to further decompose the E_6 highest weight representation into representations of its A_5 sub-algebra, indicated by nodes 6 to 10, so that the generators of the E_6 algebra may be written as A_5 tensors. This is achieved by deleting the simple root associated with node 11, which itself decomposes into a vector in the A_5 weight lattice ($-\mu_8$) and a vector orthogonal (y) as $\alpha_{11} = -\mu_8 + y$, where $\mu_{\hat{J}}$ for $\hat{J} \in \{6, 7, \dots, 10\}$ are the fundamental weights of A_5 . Consequently we have weights of A_5 , corresponding to the decomposed (internal) E_6 , such that

$$-\sum_{\hat{J}=6}^{10} r_{\hat{J}} \mu_{\hat{J}} = -m_5 \mu_6 - m_{11} \mu_8 + \sum_{\hat{J}=6}^{10} m_{\hat{J}} \alpha_{\hat{J}}. \quad (4.75)$$

By taking inner products with α_j ($j \in \{1, 2, 3, 4\}$) in (4.73) we find formulae for the coefficients p_j that label the lowest weight representations of A_4 :

$$p_1 = -2m_1 + m_2, \quad (4.76)$$

$$p_2 = m_1 - 2m_2 + m_3, \quad (4.77)$$

$$p_3 = +m_2 - 2m_3 + m_4 \quad \text{and} \quad (4.78)$$

$$p_4 = +m_3 - 2m_4 + m_5. \quad (4.79)$$

While by taking inner products with μ_{k+5} where $k \in \{1, 2, 3, 4, 5\}$ in equation (4.75) we find formulae for the coefficients m_{k+5} that label the root string subtracted from the highest weight representation of A_5 :

$$m_{k+5} = -\sum_{j=1}^5 r_{(j+5)} \frac{j(6-k)}{6} + m_5 \frac{(6-k)}{6} + m_{11} \frac{3(6-k)}{6} \quad (4.80)$$

$$= N + m_5 + 3m_{11} - (6-k) \sum_{j=1}^5 r_{(j+5)} - \frac{k}{6} (N + m_5 + 3m_{11}) \in \mathbb{Z}^+ \quad (4.81)$$

where $N \equiv \sum (6-j)r_{(j+5)}$ is the number of indices on the tensor representation of A_5 .

The bosonic content of the supermultiplet in five dimensions can be quickly reconstructed from E_{11} using the formulae for p_i and r_i above. We are interested in the scalar, vector and symmetric two-tensor representations of A_4 which correspond to $p_4 = 0, 1, 2$ respectively and $p_1 = p_2 = p_3 = 0$. From the formulae above we see that this implies $m_2 = 2m_1$, $m_3 = 3m_1$, $m_4 = 4m_1$ and hence $p_4 = -5m_1 + m_5$. So for the scalar multiplet we have $p_4 = 0$ which is satisfied by the pairs $(m_1, m_5) = \{(0, 0), (1, 5), (2, 10), \dots\}$. We will find that the scalar multiplet of five-dimensional maximal supergravity is found within the first pair $m_1 = m_2 = m_3 = m_4 = m_5 = 0$. Without any loss

of generality we find from (4.81) that $N = 3m_{11} - m_5 \geq 0$ guarantees that $m_{k+5} \in \mathbb{Z}^+$. When $m_5 = 0$ we have $N = 3m_{11}$ and we can then identify generators in A_5 associated with the scalar multiplet in A_4 for $m_{11} = 1, 2, 3$. Using

$$m_{k+5} = (6 - k)(m_{11} - \sum r_{(j+5)}) \geq 0 \quad (4.82)$$

we observe that $\sum r_{(j+5)} \geq m_{11}$. When $m_{11} = 0$ then $N = 0$ and $\sum r_{(j+5)} = 0$ so we find the Cartan sub-algebra K^M_M (**6**) and the positive generators of A_5 (K^M_N) for $N > M$ (**15**). When $m_{11} = 1$ then $N = 3 = \sum (6 - j)r_{j+5}$ and $\sum r_{(j+5)} \leq 1$ is satisfied by $r_8 = 1$ and all other $r_j = 0$ which corresponds to a three-form $R^{M[3]}$ (**20**). Finally when $m_{11} = 2$ we have $N = 6$ which is satisfied by $r_6 = 1$ while the remaining $r_j = 0$. This gives a six-form generator $R^{M[6]}$ (**1**). All other possibilities for sets of r_j are ruled out as the associated root string in A_5 has length squared greater than two. This completes the scalar multiplet having dimension **42**

$$\phi \equiv \{K^M_M(\mathbf{6}), K^M_N(\mathbf{15}), R^{M[3]}(\mathbf{20}), R^{M[6]}(\mathbf{1})\} \quad (4.83)$$

where $M_i \in \{1, 2, \dots, 6\}$ are internal A_5 tensor indices.

The origin of the five-dimensional field content is obvious from the low level generators of E_{11} . The low level E_{11} generators are

$$K^A_A, K^A_B, R^{A[3]}, R^{A[6]}, R^{A[8],B}, \dots \quad (4.84)$$

where the A and B indices are eleven-dimensional. One may quickly find the scalar multiplet of five dimensional supergravity when the dimensionally reduced generators have only internal E_6 indices. Since these internal indices are six-dimensional the $R^{A[8],B}$ generator, as well as other higher level generators, do not contribute the scalar multiplet in five dimensions.

While one could repeat the derivation of the five-dimensional scalar multiplet as initially outlined above to find the vector multiplet it is much simpler to achieve the same end by the method of partitioning the indices of the low level generators of E_{11} into internal and worldvolume indices. For the vector multiplet it will suffice to dimensionally reduce these fields so that only one of the reduced indices is a five-dimensional world-volume index. This corresponds to $p_4 = 1$ and hence $m_1 = 0, m_5 = 1$. The only possibilities amongst the low level generators are

$$\phi^\mu \equiv \{K^\mu_N(\mathbf{6}), R^{\mu M[2]}(\mathbf{15}), R^{\mu M[5]}(\mathbf{6})\} \quad (4.85)$$

where we indicate in brackets the dimension of the internal A_5 tensor representation. In total we find the **27** of the vector multiplet.

The gravitational degrees of freedom are contained in the $(\mathbf{5}, \mathbf{1})$ of $Spin(3) \otimes E_6$ which corresponds to a symmetric rank two space-time tensor carrying a trivial representation of E_6 . This representation corresponds to $p_4 = 2$ and hence $m_1 = 0$ and $m_5 = 2$. However there is no such representation appearing directly in the decomposition of E_{11} . Instead we may identify the vielbein field from which

we can construct the graviton, or we may pursue an alternative path to understand the origin of the gravitational degrees of freedom within higher level E_{11} generators. In the first instance we may identify the traceless part of the Borel sub-algebra of A_4 as the **5** of the little group in five-dimensions

$$\phi_{\text{Gravity}}^{(\mu\nu)} = \{K^\mu{}_\nu(\mathbf{5}, \mathbf{1})\} \quad \text{where } \nu \geq \mu \quad (4.86)$$

where we have indicated in brackets the $A_4 \otimes E_6$ multiplet. The fields associated with these generators, $h_\nu{}^\mu$ give the vielbein $e_\nu{}^\mu = (e^{-h})_\nu{}^\mu$ from which the graviton may be reconstructed, see section two of [35] for a detailed discussion of the vielbein within E_{11} .

Amongst the low level generators of E_{11} there is a second way the gravitational degrees of freedom may be identified using the dual graviton. Upon dimensional reduction we find a singlet of A_5 containing the five-dimensional dual graviton $R^{\mu[2]M[6],\nu}$. We note that the traceless part of $\begin{smallmatrix} \square & \square \end{smallmatrix}$ as a representation of the little group in five dimensions also gives the **5**. We may dualise the dual graviton at the linear level in its two antisymmetrised indices to a symmetric two tensor and a singlet of the internal A_5 :

$$\phi_{\text{Dual gravity}}^{(\mu\nu)} = \{\star_1 R^{\mu[2]M[6],\nu}\} \quad (4.87)$$

where $\star_1 = \iota_1 \star_H d_1$ is the action on the first set of two antisymmetric indices associated with the Hodge dual \star_H in five dimensions, d is the exterior derivative, ι is the interior product; the index on \star , ι and d indicates the set of antisymmetric indices of the mixed-symmetry tensor that the operations acts on - it indicates the column as numbered from left to right on the associated mixed-symmetry Young tableau. This is not the end of the story as amongst all the generators of E_{11} there is an infinite tower of generators each of which alone may encode the **5** gravitational sector in a similar fashion to the dual graviton. These are fields which we would naturally associate with duals of the graviton, their presence was highlighted in [15, 21] and the first of these we would suspect to be linked to Hull's double dual graviton. In five dimensions, the dual graviton $c_{\mu[2],\nu}$ and the double dual graviton $d_{\mu[2],\nu[2]}$ are related to the graviton $h_{\mu\nu}$ by $c_{\mu[2],\nu} = \star_1 h_{\mu\nu}$ and $d_{\mu[2],\nu[2]} = \star_2 c_{\mu[2],\nu} = \star_1 \star_2 h_{\mu\nu}$, the set of fields that occur naturally in E_{11} is associated with $\star_3 \star_1 h_{\mu\nu} = \bar{c}_{\lambda[3],\mu[2],\nu}$ that is the dual of a trivial "scalar". In fact we recognise these fields as the dual tower whose Young tableau are shown in (3.67), and we understand from section 2 the Curtright action may be reconciled with the double-dual action via a parent action. The relevant higher level generators of E_{11} written as eleven-dimensional tensors have the form

$$R^{A^{(1)}[9]A^{(2)}[9]...A^{(m)}[9]B[8]C} \quad (4.88)$$

for all $m \geq 0^4$. Upon reduction to five dimensions these generators are parameterised by fields which might also be interpreted as dual gravitational fields having the form

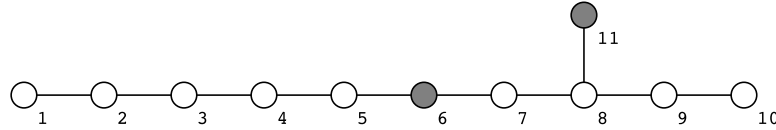
$$\phi_{(m+1)\text{-dual gravity}}^{(\mu\nu)} = \{R^{\mu^{(1)}[3]\mu^{(2)}[3]...\mu^{(m)}[3]\nu[2]\rho}\} \quad (4.89)$$

⁴when $m = 0$ we find the dual graviton.

where the internal indices, which have been suppressed, carry the singlet representation of the internal E_6 . The m sets of antisymmetric $\{\mu[3]\}$ indices transform trivially as a representation of the little group in five dimensions. Hence each of these generators is associated with fields carrying the same degrees of freedom as the traceless part of $\square\square$, the dual graviton in five dimensions. We will return to discuss this set of fields in the sequel.

4.2 Six-dimensional theories from E_{11} .

The usual lift of the five-dimensional multiplet of scalars, vectors and bivectors to six dimensions traces their origin to six-dimensional tensor objects of the same and higher rank. The setting of E_{11} does nothing to change this, but it will be useful to explicitly reproduce the lifting of the five-dimensional multiplet. The six-dimensional field content is reproduced from E_{11} by deleting nodes 6 and 11 on the E_{11} diagram as indicated below.



This results in the E_{11} generators being decomposed into representations of $A_5 \otimes A_4$, where now the A_5 corresponds to the six-dimensional space-time theory, while the A_4 representations arise from the decomposition of the internal symmetry $SO(10)$. Once again by dimensionally reducing the low level generators of E_{11} one can identify the set of six-dimensional fields which will reduce to the scalar, vector and gravity multiplet of five dimensions. The set of generators which give rise to the scalar multiplet **42** upon dimensional reduction are

$$\hat{\phi} = \{H_{\hat{\mu}}(\mathbf{1}), H_{\hat{M}}(\mathbf{5}), K^{\hat{\mu}}_{\hat{M}}(\mathbf{5}), K^{\hat{M}}_{\hat{N}}(\mathbf{10}), R^{\hat{\mu}\hat{M}[2]}(\mathbf{10}), R^{\hat{M}[3]}(\mathbf{10}), R^{\hat{\mu}\hat{M}[5]}(\mathbf{1})\} \quad (4.90)$$

where $\hat{\mu}, \hat{\nu} \in \{1, 2, \dots, 6\}$ are the six-dimensional space-time indices indicating the A_5 tensor structure and $\hat{M}, \hat{N} \in \{1, 2, \dots, 5\}$ are the five-dimensional internal indices indicating the A_4 tensor structure; H are the Cartan sub-algebra elements of E_{11} and we indicate in brackets the dimensions of the internal A_4 tensor.

Similarly we may identify the six-dimensional origin of the five-dimensional vector and gravity multiplets. The vector multiplet which transforms under the **27** of the internal symmetry is

$$\hat{\phi}^{\mu} = \{K^{\hat{\mu}}_{\hat{\nu}}(\mathbf{1}), K^{\hat{\mu}}_{\hat{M}}(\mathbf{5}), R^{\hat{\mu}[2]\hat{M}}(\mathbf{5}), R^{\hat{\mu}\hat{M}[2]}(\mathbf{10}), R^{\hat{\mu}[2]\hat{M}[4]}(\mathbf{5}), R^{\hat{\mu}\hat{M}[5]}(\mathbf{1})\}. \quad (4.91)$$

Nominally the $(\mathbf{5}, \mathbf{1})$ arises from the dimensional reduction of

$$\hat{\phi}^{(\mu\nu)} = \{K^{\hat{\mu}}_{\hat{\nu}}\} \quad (4.92)$$

although it could equally well arise from the infinite tower of fields which reduce to the five-dimensional dual graviton:

$$\hat{\phi}_{(m+1)\text{-dual gravity}}^{(\mu\nu)} = \{R^{\hat{\mu}^{(1)}[4],\hat{\mu}^{(2)}[4],\dots,\hat{\mu}^{(m)}[4],\hat{\nu}[3],\hat{\rho}]\}. \quad (4.93)$$

The internal indices transform trivially under A_4 and have been suppressed. These are the set of generators which arise from the dimensional reduction of the eleven dimensional generators of E_{11} shown in equation (4.88). Upon dimensional reduction to five dimensions these generators give those indicated in equation (4.89).

It has been argued in [13] that the strong coupling limit of the five-dimensional maximal supergravity theory is the superconformal $(4,0)$ theory in six dimensions containing **27** self-dual two-forms, **42** scalars as well as the gravitational degrees of freedom. The scalars reduce trivially to the scalars of the five-dimensional theory. The two-form which, as a representation of the little group, in six dimensions carries **6** degrees of freedom of which only **3** are independent due to the self-duality condition in six-dimensions. Upon reduction of the two-form one has a choice which degrees of freedom to use to describe the theory: either **27** two-forms or **27** vectors, which are dual to each other in five-dimensions. Instead of the Fierz–Pauli graviton, the gravitational degrees of freedom are contained in a mixed symmetry tensor $\hat{C}_{\hat{\mu}[2],\hat{\nu}[2]}$ which has the symmetries of the Young tableau:

$$\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}$$

and, on-shell, has the algebraic properties of the Weyl tensor. Because of the self-duality constraint on its curvature, the field carries only five degrees of freedom and not ten, the dimension of the Weyl tensor representation of the little group in six dimensions. Upon dimensional reduction to five dimensions there appear the graviton $h_{\mu\nu}$, the dual graviton $c_{\mu[2],\nu}$ and the double-dual graviton $d_{\mu[2],\nu[2]}$. The field \hat{C} is self-dual in six dimensions, which means that the reduced fields are not all independent. Indeed if $\hat{\star}_1 \hat{C} = \hat{C}$ (which implies $\hat{\star}_1 \hat{\star}_2 \hat{C} = \hat{C}$) where $\hat{\star}$ is derived from the Hodge dual in six dimensions, then upon reduction we have $\star_1 \star_2 d = h$ and $\star_2 d = c$ as expected for the dual and double-dual graviton. Hull referred to this as a triality relation between the three five-dimensional fields $\{h, c, d\}$, see [13].

In terms of E_{11} we understand this as a repackaging of the low-level degrees of freedom of the theory into six-dimensional scalars and tensors whose Young tableaux have columns of height two, which corresponds to a conformal sector of the theory [36]. This poses a puzzle concerning the mechanism for freezing out the other low level generators whose degrees of freedom propagate in the six-dimensional maximal supergravity. We will not pursue this here, instead we will comment upon a second problem, namely that although the graviton and the dual graviton occur directly within the decomposition of E_{11} , the double-dual graviton field $d_{\mu[2],\nu[2]}$, transforming trivially under the U-duality group, does not. Instead of the double-dual graviton in five-dimensions there is the candidate field $\bar{c}_{\mu[3],\nu[2],\rho}$ which may play the same role, and similar arguments to those presented in [13] for the

double-dual graviton and its lift to six dimensions carry across to the infinite tower of fields whose generators are shown in equation (4.89). The fields associated with these generators have Young tableaux:

$$\begin{array}{ccccc}
 \begin{array}{|c|c|} \hline n & n \\ \hline n-1 & n-1 \\ \hline \vdots & \vdots \\ \hline 4 & 4 \\ \hline 3 & 3 \\ \hline \end{array} & \dots & \begin{array}{|c|c|c|} \hline n & n & n \\ \hline n-1 & n-1 & \\ \hline \vdots & \vdots & \\ \hline 4 & 4 & \\ \hline 3 & & \\ \hline \end{array} & &
 \end{array} \quad (4.94)$$

where there are m columns of height $(n-2)$ in n dimensions. The columns of height $(n-2)$ transform trivially as a representation of the little group in n dimensions. In five dimensions when $m=1$ we have the field $\bar{c}_{\mu[3],\nu[2],\rho}$ which carries the same number of degrees of freedom as the double-dual graviton. Consider a six-dimensional field with the same symmetries $\bar{C}_{\hat{\mu}[3],\hat{\nu}[2],\hat{\rho}}$. We impose that it is self-dual in two independent ways, as $\hat{\star}_2 \bar{C} = \bar{C}$ and $\hat{\star}_1 \hat{\star}_3 \bar{C} = \bar{C}$, implying $\hat{\star}_1 \hat{\star}_2 \hat{\star}_3 \bar{C} = \bar{C}$. Upon reduction to five dimensions we find a large set of fields which are related by the six-dimensional dualities including $\bar{c}_{\mu[3],\nu[2],\rho}$ and $c_{\mu[2],\nu}$ together with $e_{\mu[2],\nu[2],\rho}$, $f_{\mu[3],\nu,\rho}$, $k_{\mu[3],\nu[2]}$, $l_{\mu[2],\nu,\rho}$, $m_{\mu[2],\nu[2]}$ and $n_{\mu[3],\nu}$. Now we note that no graviton appears directly in the reduction but $\hat{\star}_1 \hat{\star}_2 \bar{c} = h$ and $\hat{\star}_1 c = h$, so we expect a theory in which there is a choice of field used to describe the gravitational degrees of freedom. The dualities on the six-dimensional field \bar{C} reduce it to carrying **8** gravitational degrees of freedom of which **3** degrees of freedom may be eliminated by imposing that the five-dimensional field is traceless.

It is actually possible to find field equations in $6D$ for the $\bar{C}_{[3,2,1]}$ field, which yield **5** propagating degrees of freedom. Starting from the potential $\bar{C}_{\hat{\mu}[3],\hat{\nu}[2],\hat{\rho}}$, one builds the gauge-invariant curvature tensor $K_{[4,3,2]}$. All the curls of $K_{[4,3,2]}$ vanish, and we propose that the field equations set all the double traces of K to zero: $\text{Tr}^2 K_{[4,3,2]} = 0$, with the notation of [22]. These kinds of higher-trace field equations were discussed in [13, 14] and an analysis of \mathfrak{gl}_n -covariant on-shell Hodge duality for arbitrary mixed-symmetry gauge fields in Minkowski space can be found in [22]. We further impose that some of the single-traces of $K_{[4,3,2]}$ vanish, such that on-shell one has $K_{\hat{\mu}[4],\hat{\nu}[3],\hat{\rho}[2]} = \eta_{\hat{\mu}\hat{\rho}} W_{\hat{\mu}[3],\hat{\nu}[3],\hat{\rho}}$ where the tensor $W_{[3,3,1]}$ is **50**₆-irreducible. In other words, it is enough if a self-duality condition is imposed on the second column of the curvature: $*_2 K_{[4,3,2]} = K_{[4,3,2]}$. By using the general results of [22, 37], one can then show that the curvature $W_{[3,3,1]}$ is the gradient of the curvature $\hat{K}_{[3,3]}$ for the self-dual gauge field $\hat{C}_{[2,2]}$ discussed by Hull [13] and introduced in [34]. Note that, by using the equations $K_{\hat{\mu}[4],\hat{\nu}[3],\hat{\rho}[2]} = \eta_{\hat{\mu}\hat{\rho}} W_{\hat{\mu}[3],\hat{\nu}[3],\hat{\rho}}$ for a traceless $W_{[3,3,1]}$, the relation $\hat{\star}_1 \hat{\star}_3 \bar{C} = \bar{C}$ proposed above is indeed satisfied. With the above field equations, the $\bar{C}_{[3,2,1]}$ field in six-dimension propagates the same **5** degrees of freedom as does $\hat{C}_{[2,2]}$, which is what we wanted to show.

We conclude this section with some comments on the construction of Young tableaux of E_{11} generators, which will be analogous to the auxiliary Z fields appearing in the earlier sections of this note, required for a covariant formulation of the double-dual graviton. Generalised Kac-Moody

algebras are constructed from their Cartan matrix A together with the Serre relations:

$$\overbrace{[E_a, [E_a, \dots [E_a, E_b] \dots]]}^{1-A_{ab}} = 0 \quad \text{and} \quad \overbrace{[F_a, [F_a, \dots [F_a, F_b] \dots]]}^{1-A_{ab}} = 0 \quad (4.95)$$

where $E_a(F_a)$ are the positive(negative) generators of the algebra, A_{ab} is a Cartan matrix entry and there are $(1 - A_{ab})$ E_a or F_a generators in each relation. These relations give constraints⁵ on the root length of the roots associated with the algebra which may be directly related to the Young tableaux of the generators of the algebra [38]. For E_{11} a generic root $\vec{\beta} = \sum_{i=1}^{11} w_i \vec{e}_i$ is associated with a generator whose Young tableau has rows of width w_i . The root length squared of this root is

$$\beta^2 = \sum_{i=1}^{11} (w_i)^2 - L^2 \quad (4.96)$$

where $L \equiv \frac{1}{3} \sum_{i=1}^{11} w_i$ is the level the generator appears at in the decomposition of E_{11} and is one-third of the number of boxes in its Young tableau. The root length formula is such that if one moves a single box of a mixed-symmetry Young tableau one column to the left the root length squared is reduced by two. Suppose that one moves a single box in this way, this corresponds to $w_k \rightarrow w_k - 1$ and $w_l \rightarrow w_l + 1$ (for some row k and another row l) and $L \rightarrow L$. The root length squared changes as

$$\beta^2 \rightarrow \beta'^2 = \sum_{i=1}^{11} (w_i)^2 - 2w_k + 2w_l + 2 - L^2 = \beta^2 - 2 \quad (4.97)$$

where we have used the observation that $w_k = w_l + 2$ since the box is moved from one column to the top of the adjacent column to the left. This is a useful observation as given a real root associated with a mixed symmetry Young tableau one can identify a sequence of null and imaginary roots in the algebra whose generators have the symmetries of Young tableaux formed by repeatedly moving boxes to the left. For example, the five-dimensional field $\bar{c}_{\mu[3]\nu[2]\rho}$ is associated with a real root appearing at level six in the decomposition of E_{11} , while at the same level there appear null and imaginary roots also transforming trivially under the internal E_6 symmetry, which are derived by moving boxes to the left in the Young tableau as shown in table 1.

This is the same pattern that occurred in the gauge fields used to construct the action for the dual graviton tower of gravitational degrees of freedom highlighted in section 3. The decomposition of the first of these fields is shown in equation (3.68), where the off-shell gauge fields required to guarantee that the Curtright field propagates the correct number of gravitational degrees of freedom are shown. It is not a coincidence that the sets of fields identified within E_{11} and those required to constrain the off-shell degrees of freedom are the same. In both cases it is the same consideration of identifying the irreducible highest weight representations that singles out the sets of fields. So we expect the gauge

⁵Roots of E_{11} have length squared which is bounded from above and if this is normalised to two then $\beta^2 = 2, 0, -2, -4, \dots$






Root length squared, β^2	Field	Multiplicity	Outer multiplicity, μ
2	 $\bar{c}_{\mu[3],\nu[2],\rho}$	1	1
0	 $q_{\mu[4],\nu,\rho}$	8	1
0	 $r_{\mu[3],\nu[3]}$	8	1
-2	 $s_{\mu[4],\nu[2]}$	44	4
-4	 $t_{\mu[5]\nu}$	206	5

Table 1: The fields of E_{11} reduced to five dimensions with six tensor indices which transform trivially under the internal E_6 symmetry.

fields required to construct the Curtright action for the tower of dual graviton fields to match the fields which appear in E_{11} at the same level as the multi-dual graviton generator. This leads to the tantalising possibility that null and imaginary roots of Kac-Moody algebras may be associated with gauge fields, and gauge for gauge fields and so on.

5 Conclusion

In this paper we constructed an explicit linearised action for the double-dual graviton given in equations (2.50-2.53) for arbitrary dimension, $n \geq 5$. The off-shell Hodge dualisation that transforms the linearised graviton $h_{\mu\nu}$ into the dual graviton $C_{\mu[n-3],\nu}$ and the double-dual graviton $D_{\mu[n-3],\nu[n-3]}$ were understood at the level of their corresponding actions. This was done by first constructing a parent action $S[\Omega, Y]$ where the algebraic elimination of one set of fields reduced the parent action to the Curtright action but by eliminating the other set of fields the action for the double-dual graviton remained. This procedure had previously been used to construct a parent action which related the

Fierz–Pauli action to the Curtright action [6]. We indicate the series of parent actions and their algebraic reductions by the sketch below.

$$\begin{array}{ccc}
S[\Omega_{a[2],b}, Y_{a[3],b}] & & S[H_{a[n-3],b[2]}, D_{b[3],a[n-3]}] \\
\swarrow & \searrow & \swarrow \quad \searrow \\
S_{\text{FP}}(h_{\mu\nu}) & S_{\text{Curt.}}(C_{\mu[n-3],\nu}) & S_{\text{DD}}(D_{\mu[n-3],\nu[n-3]})
\end{array}$$

While the off-shell dualisation trivially transforms the fundamental gravity field the transformation of the actions is not so simple and an increasingly complicated set of auxiliary fields are required to construct the action for each subsequent dualisation. The set of fields required to guarantee that the double-dual graviton propagates the correct number of degrees of freedom is given in equation (2.24). There are two novel aspects that appear in the construction of the action for the double-dual graviton.

- (i.) The double-dual action retains a number of mixed-symmetry fields on top of the \mathfrak{gl}_n -irreducible $[n-3, n-3]$ field. These extra fields are required for consistency of the action principle and correct number of propagating degrees of freedom. They are not auxiliary in the sense that their equations of motion cannot be solved algebraically in terms of the $[n-3, n-3]$ field, however they can be eliminated on-shell by differential gauge symmetries. The detailed mechanism will be explained elsewhere [39] using the frame formulation. Such a situation did not occur for the parent actions of [6].
- (ii.) The action retains an unfixed parameter β , which corresponds to the possibility of adding total derivative terms to the Curtright Lagrangian when expressed in terms of the curl $H_{[n-3,2]}(C)$. The relation between two different child actions can be viewed as a Legendre transformation. At certain values of β , the Legendre transformation becomes non-invertible. This is what happens for the two values $\beta^{(1)}$ and $\beta^{(2)}$ given after (2.53). Fixing a non-singular value for β in (2.41) will correspondingly fix it in the Lagrangian of (2.59).

In addition to the double-dual graviton there exists an infinite tower of dual fields each carrying the gravitational degrees of freedom, this was observed in section 3. These infinite towers carry representations of the little group $SO(n-2)$ in n dimensions which are non-trivial and all equivalent to the spin-2 $SO(n-2)$ -irrep through multiplication/contraction by a number of $SO(n-2)$ metric tensors and antisymmetric $SO(n-2)$ symbols. Correspondingly, the \mathfrak{gl}_n -covariant field equations will feature high powers of the trace operations on the curvature tensor K , as was proposed in [14, 22].⁶ In this context, the recent paper [40] also considered the \mathfrak{g}^{+++} tower of fields obtained by attaching columns of length $(n-2)$ to the Young diagrams associated with a finite-dimensional algebra \mathfrak{g} , and under some assumptions concluded that these fields carry no propagating degrees of freedom.

⁶The double-dual graviton introduced by Hull [12, 14] is the prototype for such a situation and partly motivated the general analysis presented in [22].

We think that the present analysis might be relevant in this context, as we have seen, for example, that the double-dual graviton possesses, as a subset of its complete set of gauge parameters, those of a type- $[n-3, n-3]$ Labastida-like gauge field. That covariant equations of motion relevant for mixed-symmetry gauge fields can bring in higher powers of the traces of the curvature tensors⁷ is a feature that might be taken into account in trying to build an E_{11} -invariant off-shell formulation of supergravity. In this work we did not investigate the possibility of building an action invariant under E_{11} and such that only one graviton would propagate on-shell in the spin-2 sector, but we hope that our results can be useful for that goal. In any case, our results show that the Labastida formulation is not the only relevant one for such a purpose.

We characterise three towers of fields as the Fierz–Pauli tower shown in equation (3.65) which is derived from the linearised graviton, the dual graviton tower shown in equation (3.67) which is derived from the dual graviton and the double-dual tower shown in equation (3.69) which is derived from the double-dual graviton. The dual graviton tower was first recognised as an infinite set of dual graviton fields in [15] where they were identified within the algebra of E_{11} . In this paper we presented the steps required to construct the linearised action associated with any of the fields in each of the three gravity towers. The set of fields required to write down the off-shell action grows with the number of dualisations needed to relate the field to the graviton. The actions, although they propagate the same degrees of freedom as the graviton, contain many more fields. In [15] towers of fields dual to the membrane and the fivebrane gauge fields of eleven-dimensional supergravity were also found within E_{11} and dual actions can be constructed in the same manner for these infinite sets of fields as for the dual graviton tower using the prescription given in this paper.

The work presented here was, in part, inspired by the argument that the strong coupling limit of five dimensional maximal supergravity contains the double-dual graviton [13]. It was expected that the corner of M-theory identified by the strong coupling limit with a six-dimensional $(4,0)$ superconformal theory would be discovered within E_{11} . However no such double-dual graviton is contained within E_{11} , instead the dual graviton tower of multi-dual graviton fields [15] are the only candidate dual gravity fields. We showed in section 4 that first of the dual gravity fields, $\bar{c}_{\mu[3]\nu[2]\rho}$ in five dimensions, whose symmetries are described by the Young tableau $\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \end{array}$ propagate the same **5** degrees of freedom as the double-dual graviton $\bar{d}_{\mu[2]\nu[2]}$ whose symmetries are those of the Young tableau $\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}$. Indeed $\bar{c}_{\mu[3]\nu[2]\rho}$ is singled out as it transforms trivially under the internal E_6 symmetry (as seen from the decomposition of E_{11} to five dimensions). We noted on-shell that the gauge field might have a number of dualities imposed upon it namely $\bar{C}_{[3,2,1]}$ satisfy $\hat{\star}_2 \bar{C} = \bar{C}$, $\hat{\star}_1 \hat{\star}_3 \bar{C} = \bar{C}$ and $\hat{\star}_1 \hat{\star}_2 \hat{\star}_3 \bar{C} = \bar{C}$. The requirement


⁷We note that the formulation $\text{Tr } K = 0$ of the Labastida equations was found in [41]. It is thanks to this reformulation of the Labastida equations that it became possible to prove that the Labastida equations are actually correct, ensuring unitarity and absence of ghosts, see [37] and detailed discussions therein.


that the following six-dimensional field equations are imposed

$$\text{Tr}^2 K_{[4,3,2]} = 0 \quad , \quad K_{\hat{\mu}[4],\hat{\nu}[3],\hat{\rho}[2]} = \eta_{\hat{\mu}\hat{\rho}} W_{\hat{\mu}[3],\hat{\nu}[3],\hat{\rho}} \quad \text{and} \quad K_{[4,3,2]} = *_2 K_{[4,3,2]} \quad , \quad (5.98)$$

where $K_{[4,3,2]}$ is the field strength [14, 22] of $\bar{C}_{[3,2,1]}$, is sufficient to ensure that the strong coupling lift of $\bar{c}_{[3,2,1]}$ to $\bar{C}_{[3,2,1]}$ in six dimensions preserves the E_6 multiplets as well as the gravitational degrees of freedom. These are similar to the arguments which were made by Hull [13] for the double-dual graviton.

It was emphasised above that the linearised action for the double-dual graviton retained a number of supplementary mixed-symmetry fields. These mixed symmetry fields were identified in the construction of the action from the irreducible components of double-dual graviton which could not be completely gauged away by algebraic gauge transformations. The same is true of the first field in the

dual graviton tower of dualities, namely in five-dimensions the  and a set of mixed symmetry fields expected to be retained in the corresponding action are listed in equation (3.68). The construction of E_{11} rests upon the Serre relations, see equation (4.95), which guarantee the irreducibility of any representation of E_{11} . The irreducibility restricts which mixed-symmetry tensors appear in the decomposition of E_{11} to tensor representations of \mathfrak{sl}_{11-n} relevant to n dimensional extended supergravity.

It is not coincidental that the same sets of mixed symmetry fields required for the action of the  in five dimensions are also contained within E_{11} . It is the same consideration of irreducibility that has been used both in the programme for constructing dual actions and in the definition of E_{11} . The corresponding supplementary mixed-symmetry tensors are associated with null and imaginary roots in the root system of E_{11} .

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